A SHADOWING-TYPE DATA ASSIMILATION METHOD FOR PARTIALLY OBSERVED SYSTEMS

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SHADOWING LEMMA

Let ϕ^{t_n} be a flow (e.g. numerical discretisation) associated with a continuous dynamical system $\dot{z} = f(z)$:

$$v_{n+1} = \phi^{t_n}(v_n), \quad for \quad n = 0,\ldots,N-1, \quad where\ v_n \in \mathscr{R}^m$$

Shadowing lemma (A. Katok and B. Hasselblatt, 1995): There exists the true orbit $\{u_n^{\text{true}}\}_{n=0}^N$ with $u_{n+1}^{\text{true}} = \phi^{t_n}(u_n^{\text{true}})$, such that

$$||u_n - u_n^{\text{true}}|| < \delta, \quad for \quad n = 0,\ldots,N-1$$

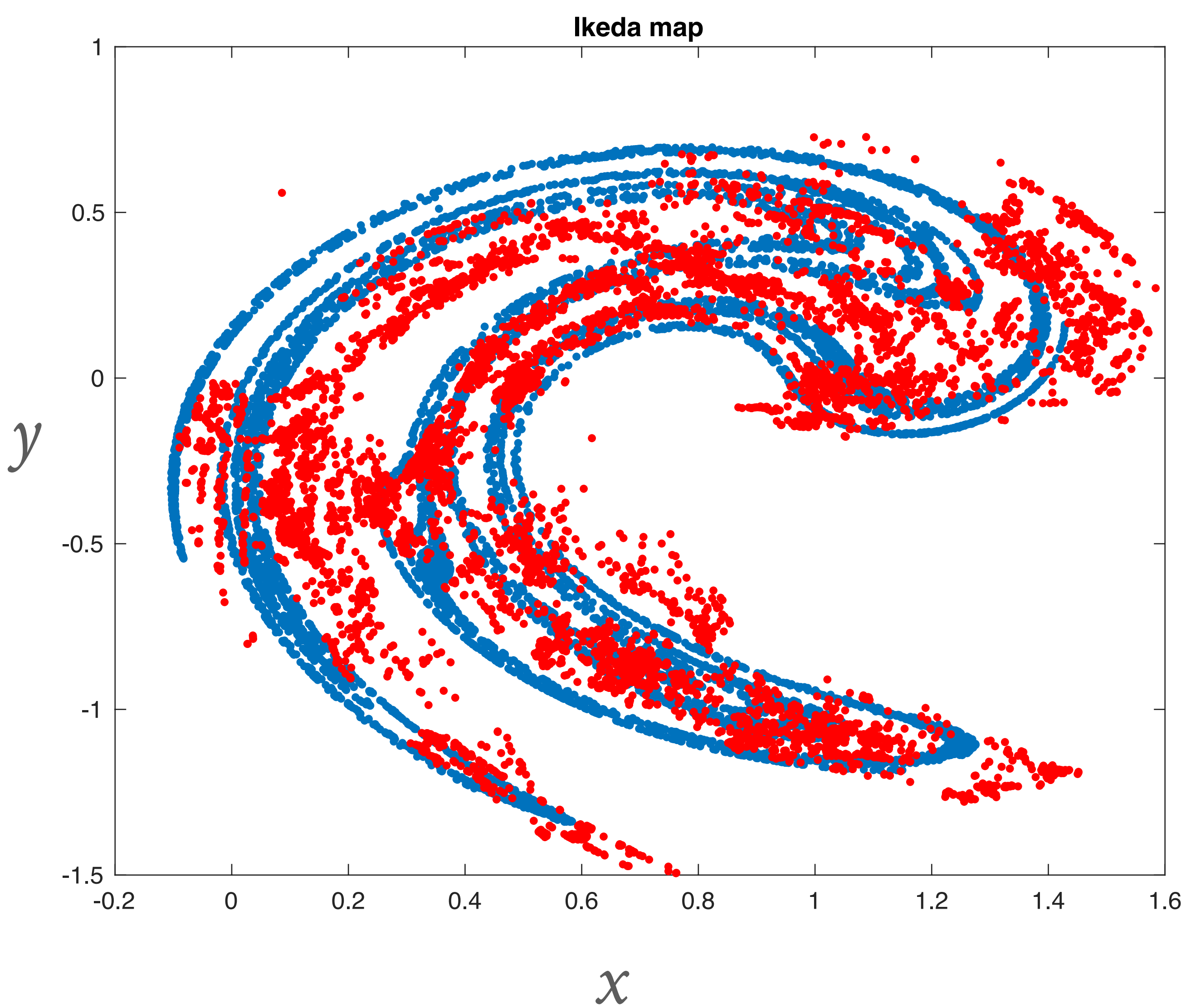
where $\{u\}_{n=0}^N$ is an ε-pseudo-orbit, namely

$$||u_{n+1} - \phi^{t_n}(u_n)|| < \epsilon, \quad for \quad n = 0,\ldots,N-1$$

The Shadowing lemma guarantees the existence of a solution in a δ-neighbourhood of $\{u_n^{\text{true}}\}_{n=0}^N$

SHADOWING APPROACH TO DATA ASSIMILATION

"The principle idea of shadowing-based data assimilation is to take observations of a trajectory (red dots) and to relax these onto a near-by trajectory (blue dots)." *K. Judd and L. Smith (2001)*.



K. Judd and L. Smith (2001)

J. Brocker and U. Parlitz (2001)

K. Judd et al. (2008)

T. Stemler and K. Judd (2009)

H. Du and L. Smith (2014)

ITERATIVE METHODS FOR SHADOWING

Define the function G as

$$G_n := u_{n+1} - \phi^{t_n}(u_n)$$

Find zeros of G by an iterative method

$$u^{(j+1)} = u^{(j)} + \Delta^{(j)}$$

Initiate the method at full observations

$u^{(0)} = y,$

$y_k = u_k^{\text{true}} + \xi_k, \quad \text{for} \quad 0 \leq k \leq N-1, \quad \text{where} \quad \xi_k \sim \mathcal{N}(0, R)$

$u_{n+1}^{\text{true}} = \phi^{t_n}(u_n^{\text{true}}), \quad \text{for} \quad n = 0, \ldots, N-1$

EXISTING SHADOWING-BASED DA METHODS

$$u^{(j+1)} = u^{(j)} + \Delta^{(j)}, \qquad P := G'(u^{(j)})$$

1) $\Delta^{(j)} = -\gamma P^T G(u^{(j)})$ *Judd and Smith (2001); Du and Smith (2014)*

2) $\Delta^{(j)} = -P^T \Lambda^{-1} G(u^{(j)})$ *Brocker and Parlitz (2001)*

3) $\Delta^{(j)} = -P^T (PP^T)^{-1} G(u^{(j)})$ *de Leeuw et al. (2018)*

All these methods are initiated at a (proxy of) of **full** observations.

SHADOWING-BASED DA FOR PARTIAL OBSERVATIONS

We use a regularized Gauss-Newton method to find a pseudo-orbit

$$u^{(j+1)} = u^{(j)} + \Delta^{(j)}, \quad \textit{where}$$

$$\Delta^{(j)} = -\,\Sigma P^T (P \Sigma P^T + \alpha Q)^{-1} G(u^{(j)}) \quad \textit{and} \quad P := G'(u^{(j)})$$

The initial guess $u^{(0)} = \mathscr{Y}$ consists of partial observations y and a background trajectory—a model trajectory started from an arbitrary initial guess.

Y. Chen and D. Oliver (2013);
Ebtehaj, A. M., M. Zupanski, G. Lerman, and E. Foufoula-Georgiou (2014)
de Leeuw and S.D. (2022)

LOCAL CONVERGENCE AND TRUST REGION

- Theorem I: Under some conditions on the initial guess and a regularization parameter α, the shadowing-based DA method converges locally to the solution manifold
  $||u_{n+1} - \phi^{t_n}(u_n)|| < \epsilon, \quad \text{for} \quad n = 0, \ldots, N-1$

- Theorem II: Under some conditions, a shadowing-based estimate projected on the observation space remains in a ball centred at the observations and radius of the observation error.

  In practice: in order to fulfil the conditions of Theorem II, we need to choose a specific preconditioning Σ for the Gauss-Newton method

$$\Delta^{(j)} = -\Sigma P^T(P\Sigma P^T + \alpha Q)^{-1}G(u^{(j)}) \quad \textit{and} \quad P := G'(u^{(j)})$$

EXISTING SHADOWING-BASED DA METHODS

$$u^{(j+1)} = u^{(j)} + \Delta^{(j)}, \qquad P := G'(u^{(j)})$$

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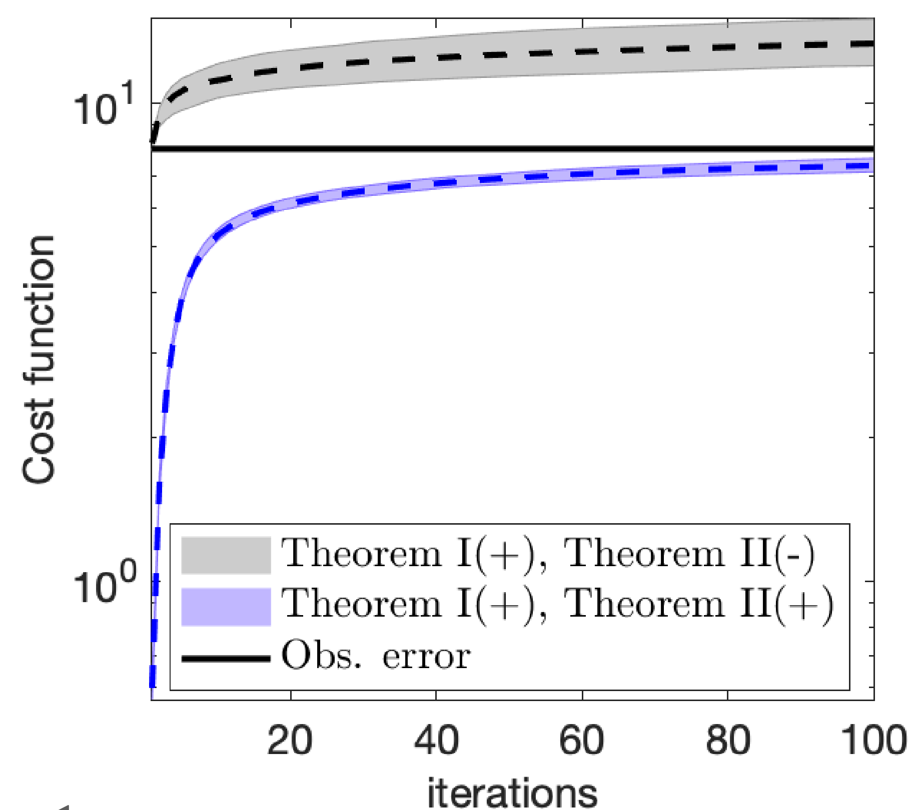
4) $\Delta^{(j)} = -\Sigma P^T (P \Sigma P^T + \alpha Q)^{-1} G(u^{(j)})$ *de Leeuw and S.D. (2022)*

All the methods (1)—(4) converge to the solution manifold.

Choosing an appropriate Σ in (4) leads to a good estimation of the true solution.

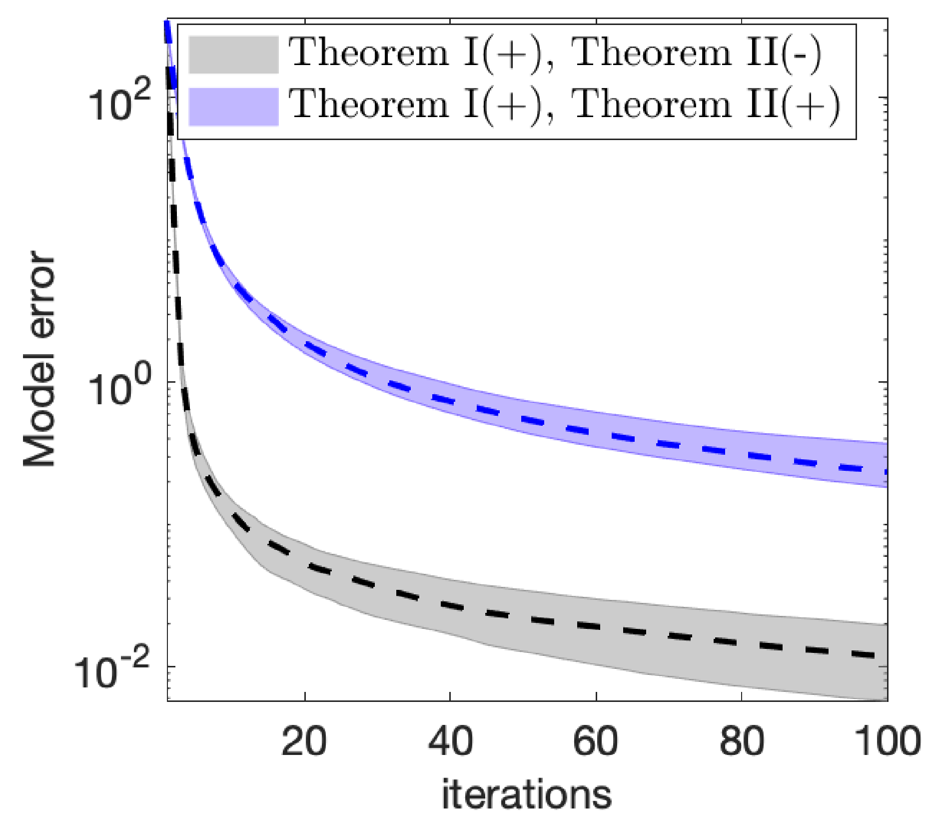
THE SHADOWING-BASED DA METHOD WITH PARTIAL OBSERVATIONS: NUMERICAL EXPERIMENT

We observe every 2nd variable of the Lorenz 96 model every 6 hours over 25 days. Variance of the observation error is 8.



$$\frac{1}{\#K}\sum_{k\in K}(Hu_k - y_k)^T(Hu_k - y_k)$$

H is the observation operator that projects an estimate onto the observation phase space



$$\frac{1}{N}\sum_{n=0}^{N-1} G_n^T G_n$$

$$G_n := u_{n+1} - \phi^{t_n}(u_n)$$

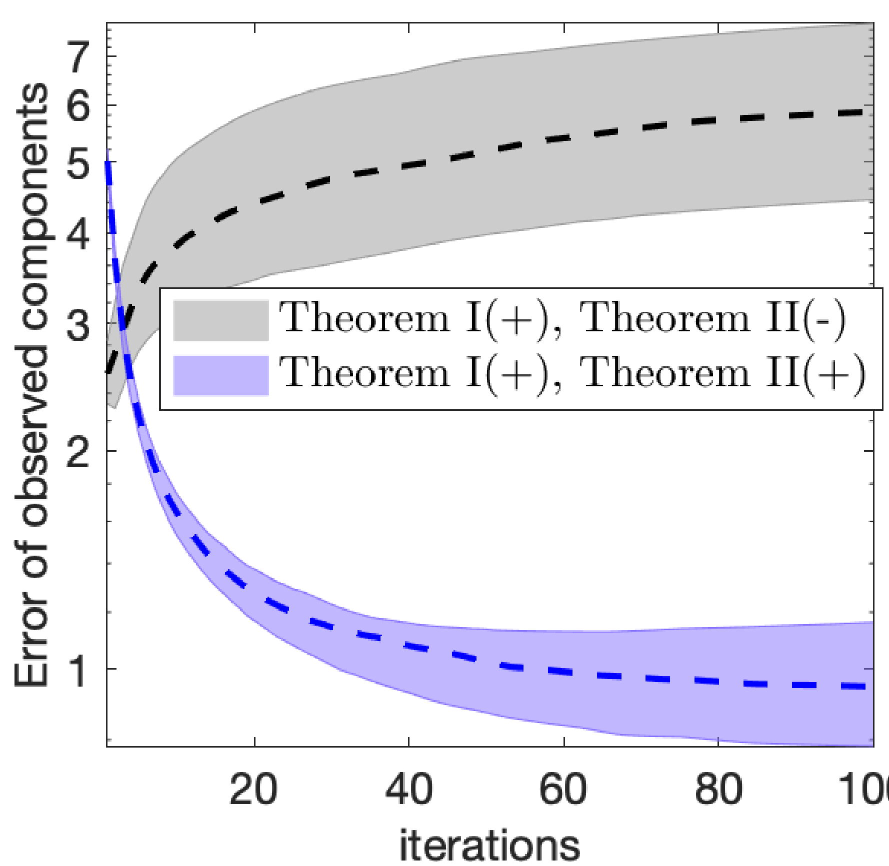
ERROR WITH RESPECT TO THE TRUE SOLUTION

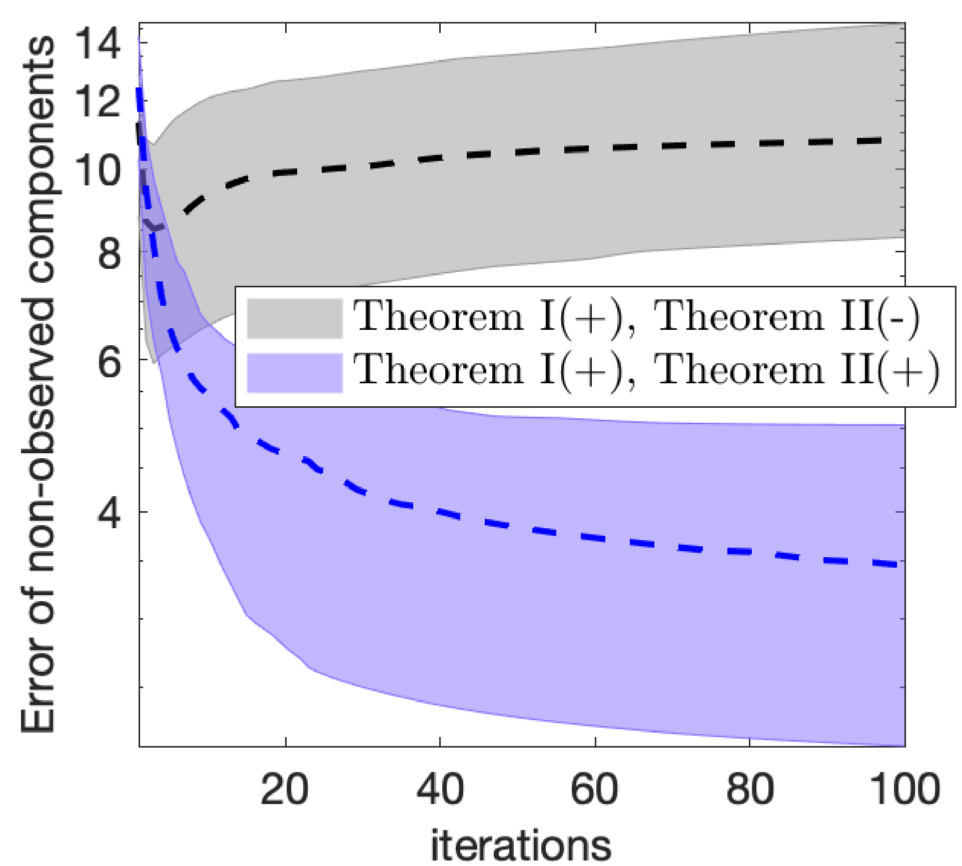
$$\frac{1}{N}\sum_{n=0}^{N-1}(Hu_n - Hu_n^{\text{true}})^T(Hu_n - Hu_n^{\text{true}})$$

H is the observation operator that projects an estimate onto the observation phase space

$$\frac{1}{N}\sum_{n=0}^{N-1}(H^{\perp}u_n - H^{\perp}u_n^{\text{true}})^T(H^{\perp}u_n - H^{\perp}u_n^{\text{true}})$$

$H^{\perp}$ is an operator that projects an estimate onto the “non-observed” phase space

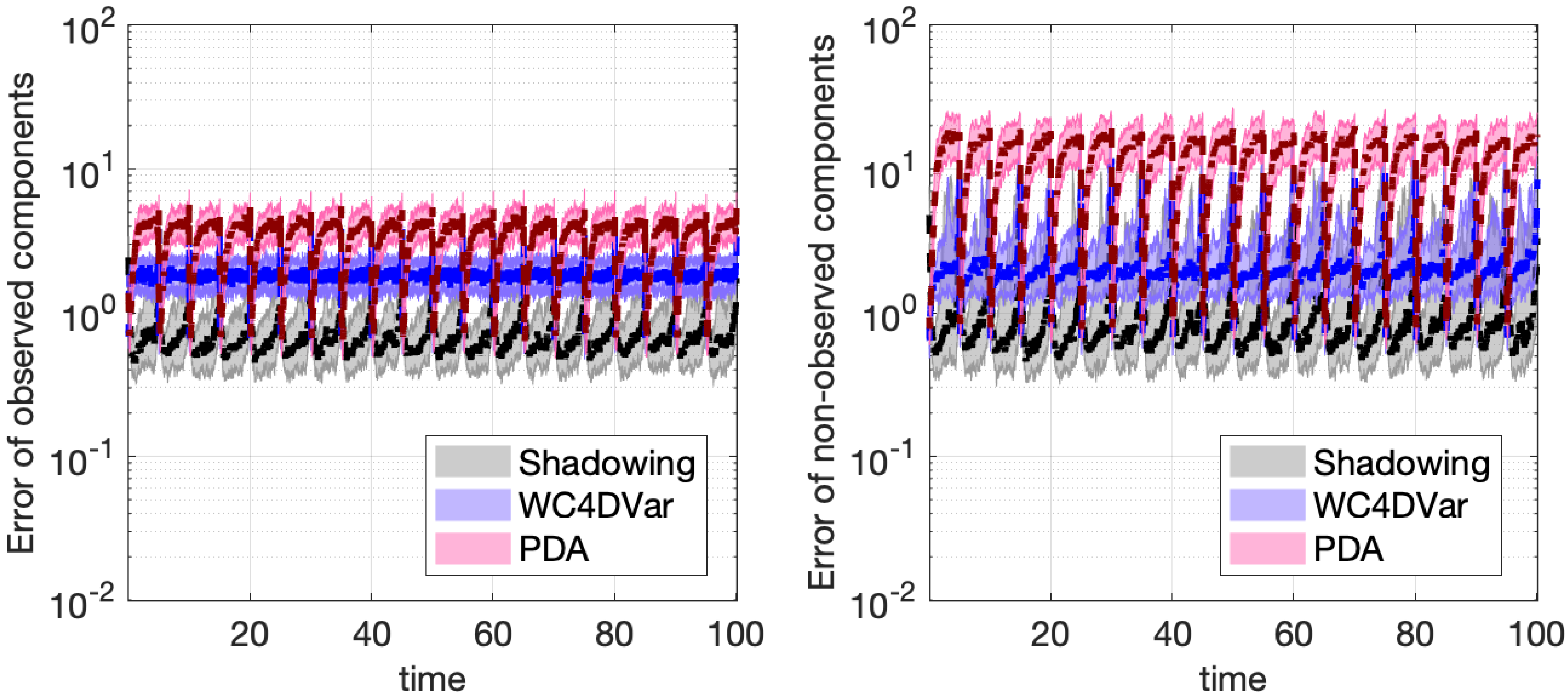




COMPARISON TO OTHER DA METHODS

We compare the shadowing-based DA method to a weak constraint variational method and to a Pseudo-orbit DA method.

We plot error with respect to the true solution over time.



ASSIMILATION IN THE UNSTABLE SUBSPACE

- Recent efforts to improve speed and reliability of data assimilation specifically address the partitioning of the tangent space into stable, neutral, and unstable subspaces corresponding to Lyapunov vectors associated with negative, zero, and positive Lyapunov exponents, respectively: 4DVAR-AUS, projected ensemble Kalman filter

A. Trevisan, M. D'Isidoro, and O. Talagrand (2010); L. Palatella, A. Carrassi, and A. Trevisan (2013); C. Gonzalez-Tokman and B. R. Hunt (2013); K. J. H. Law, D. Sanz-Alonso, A. Shukla and A. M. Stuart (2016)

- A dimension of the unstable subspace is smaller than a dimension of the model: 24 vs 14724 for a QG model (*R. Rotunno and J.-W. Bao 1996*)

PROJECTED SHADOWING-BASED DA METHOD

Motivated by these works, we propose a new method for shadowing-based data assimilation that utilises distinct treatments of the dynamics in the stable and nonstable (neutral and unstable) directions *(B. de Leeuw et al, 2018)*.

Novel projected shadowing-based DA method:

- We construct projection operators onto the stable and nonstable subspaces.
- In the nonstable subspace, we perform (expensive) shadowing-based DA that gives us a very accurate estimate.
- In the stable subspace, we decrease error by means of synchronisation to that accurate estimate.

SYNCHRONISATION IN DATA ASSIMILATION

Research on synchronisation of chaos indicates that

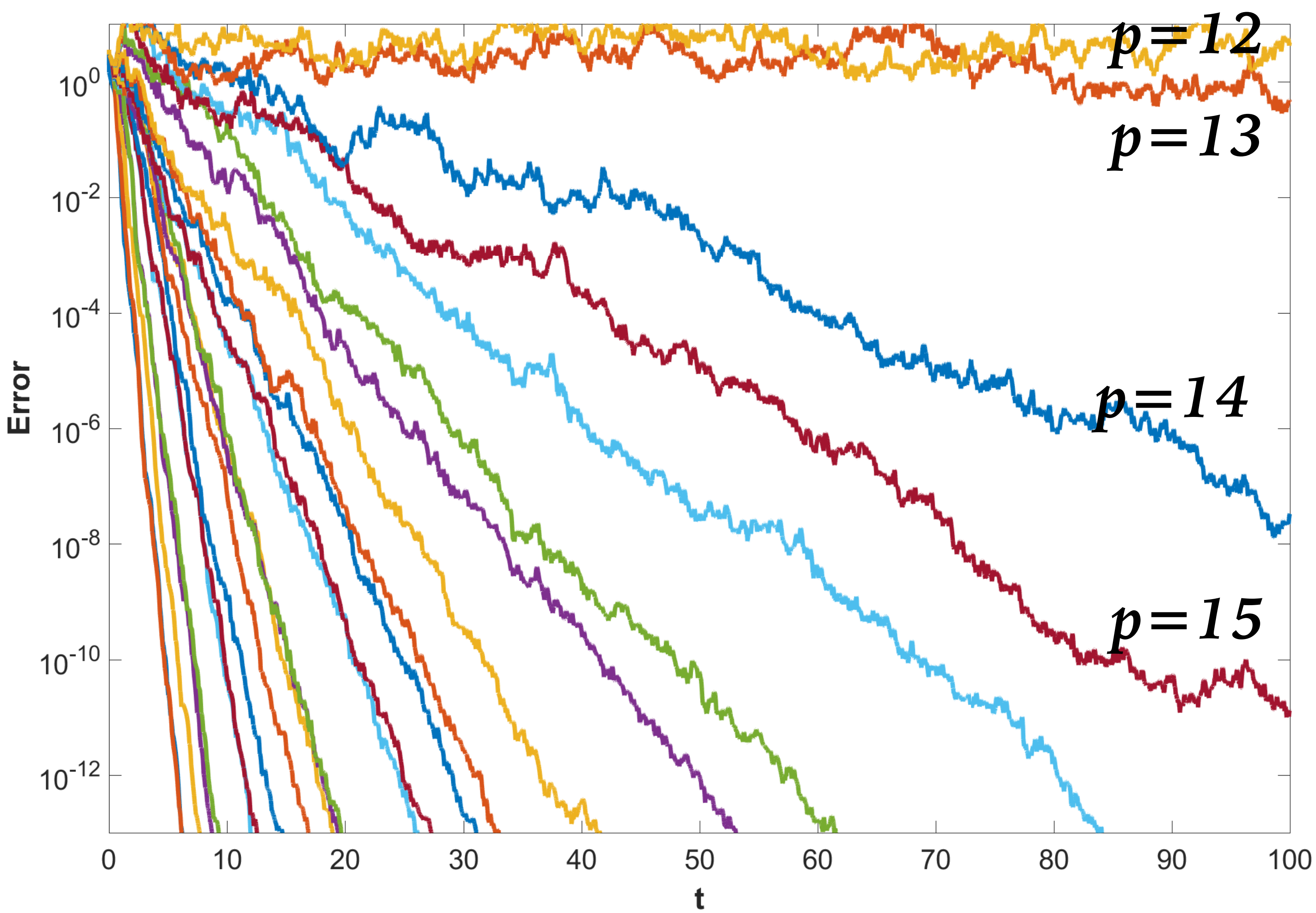
- when partial observations are sufficient to constrain the unstable subspace,
- an orbit of a chaotic dynamical system can be made to converge exponentially in time to a different, driving orbit.

(provided exponential dichotomy)

Pecora and Carroll (1990);
Pecora et al. (1997);
Boccaletti et al. (2002)

SYNCHRONISATION OF THE LORENZ 96 MODEL

- We consider the Lorenz 96 model (36 variables). It has 13 positive Lyapunov exponents.
- The true solution is partially observed (noise free): we have access to the true solution projected onto the non-strongly stable subspace of dimension p.
- Note that the dimension of the nonstable subspace is 14.



We plot

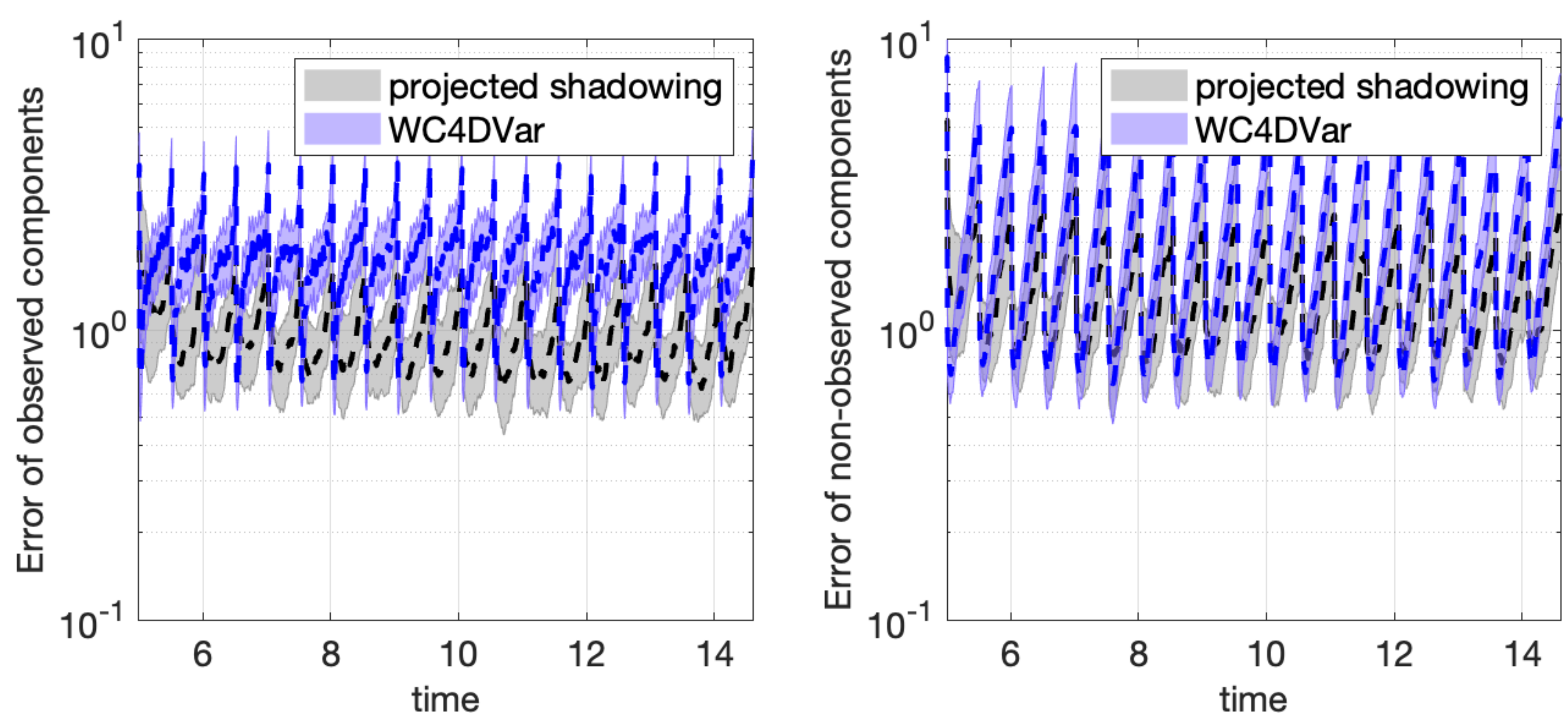
- *the difference between the true solution and the synchronisation approximation in the infinity norm*
- *as a function of time*
- *for different p*

NUMERICAL EXPERIMENT

We compare the **projected** shadowing-based DA method to a weak constraint variational method.

We consider the Lorenz 96 model. The projection dimension is 25.

We plot error with respect to the true solution over time.



CONCLUSIONS

- We have developed a shadowing-based DA method specifically for partial observations.
- The method converges to the solution manifold. Moreover, the solution projected onto the observation space is within a ball centred at the observation with radius of the observation error.
- We extended the method to nonstable subspace.
- We have shown numerically that the (projected) shadowing-based DA method provides a good estimation of the true solution. Moreover, it outperforms both WC4DVar and PDA.

B. de Leeuw and S.D., "Shadowing-based data assimilation method for partially observed models", SIADS (2022).

Thank you

for

your attention!