Hybrid data assimilation for hierarchical models[a]

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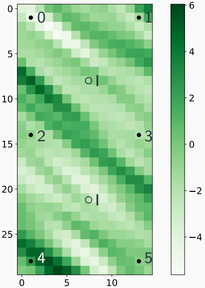
2 June 2022

[a] "Hybrid" is probably overused – this is an ensemble Kalman-like method with some analytical gradients (hence hybrid).

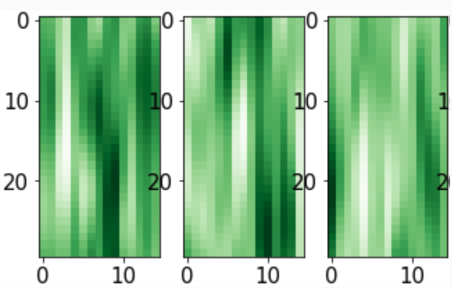
Parameter estimation

- Estimate the spatially distributed coefficients of the pde for flow and transport in porous medium from observations of rates at well locations (e.g. inject water to displace oil, inject CO_2 for sequestration, transport of pollutants in groundwater).
- Need to specify a prior pdf for model parameters – often use Gaussian.
- Fixing hyperparameters of the prior at incorrect values may result in failure to assimilate data and failure to adequately represent uncertainty.

Consequence of bad prior model



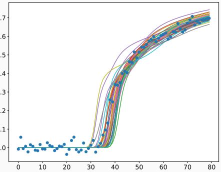
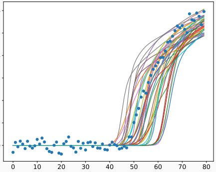
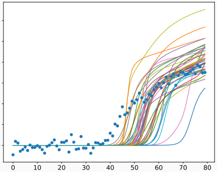
true ln K



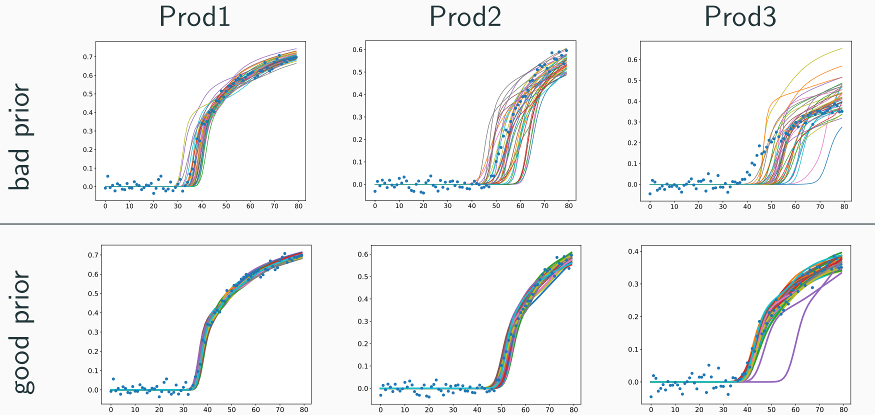
realizations from “bad” prior

Two injectors and six producers. All rates fixed. Only measure water cut.

Consequence of bad prior model – posterior predictive distribution

	Prod1	Prod2	Prod3
bad prior			

Consequence of bad prior model – posterior predictive distribution



Only difference in the priors is the orientation of the anisotropy.

Much better predictability if the prior is chosen appropriately.

Geometric anisotropy[1]

Assume stationarity and geometric anisotropy

$$\mathrm{cov}(m_x, m_{x'}) = \sigma^2 f\left(\frac{(x-x')^{\mathrm{T}} A^{\mathrm{T}} A (x-x')}{\rho^2}\right)$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$$

[1] Shen and Gelfand (2019)

Non-centered reparameterization[2]

Allow uncertainty in the prior covariance.

Instead of using

$$m, \quad \overbrace{\phi, \rho, \alpha}^{\text{hyperparameters}}$$

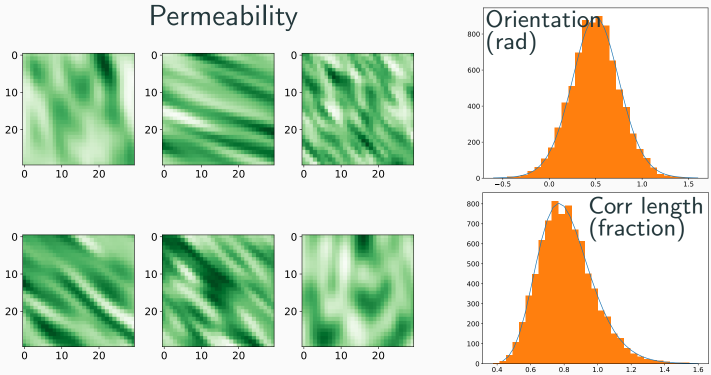
where m are our usual parameters, we use

$$z, \quad \overbrace{\phi, \rho, \alpha}^{\text{hyperparameters}}$$

where $m = m_{\mathrm{pr}} + C_m^{1/2} z$

m are physical model parameters, z are standard normal iid random variables.

[2] Papaspiliopoulos et al. (2003); Chada et al. (2018)



The initial ensemble of realizations can include many different orientations and correlation ranges.

Data assimilation

Posterior pdf for non-centered parameterization

$$p(x|d) \propto \exp\left(-\frac{1}{2}(d^o - g(m(x)))^{\mathrm{T}} C_d^{-1}(d^o - g(m(x)))\right) \times \exp\left(-\frac{1}{2}(x-\bar{x})^{\mathrm{T}} C_x^{-1}(x-\bar{x})\right).$$

Approximate sampling from posterior by minimization of cost function

$$x^{\mathrm{post}} = \underset{x}{\operatorname{argmin}} \|d^o - g(m(x)) - \epsilon^*\|^2_{C_d^{-1}} + \|x - x^*\|^2_{C_x^{-1}}.$$

where $\epsilon^* \sim N[0, C_d]$ and $x^* \sim N[\bar{x}, C_x]$.

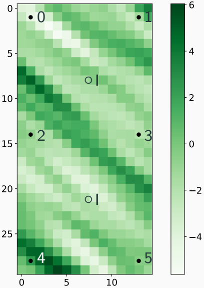
For history matching, we may use an IES to approximately sample:

$$
\begin{aligned}
\delta x_{\ell+1} = -\Delta x_\ell \, \Delta x_\ell^{\mathrm{T}} C_x^{-1} (x_\ell - x^*) \\
- \overbrace{\Delta x_\ell \, \Delta d_\ell^{\mathrm{T}} \left(C_d + \Delta d_\ell \Delta d_\ell^{\mathrm{T}} \right)^{-1}}^{\text{“Kalman gain matrix”}} \\
\times \left(g(m_\ell) + \epsilon^* - d^{\mathrm{obs}} - \Delta d_\ell \Delta x_\ell^{\mathrm{T}} C_x^{-1} (x_\ell - x^*) \right)
\end{aligned}
$$

where $x = (z, \phi, \rho, \alpha)$ and $\Delta x_\ell = \frac{(X_\ell - \bar{X}_\ell)}{\sqrt{(N-1)}}$ and similar for Δd_ℓ.

Fluid flow example

Model and data



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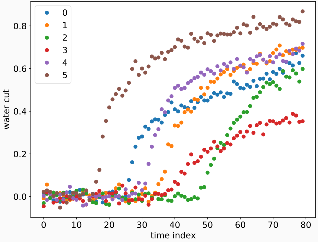
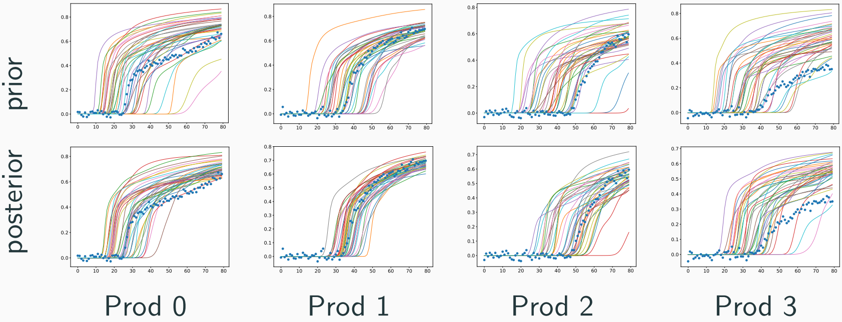


observations

Two injectors and six producers. All rates fixed. Only measure water cut.

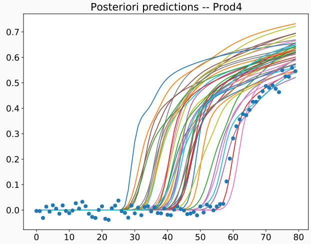
True permeability field is anisotropic.

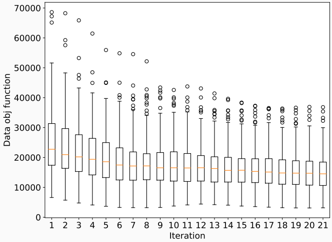
History matching hierarchical model with standard IES



Improvement from history matching with IES is small. Iterative ensemble filter does not work well on this application.

History matching hierarchical model with standard IES



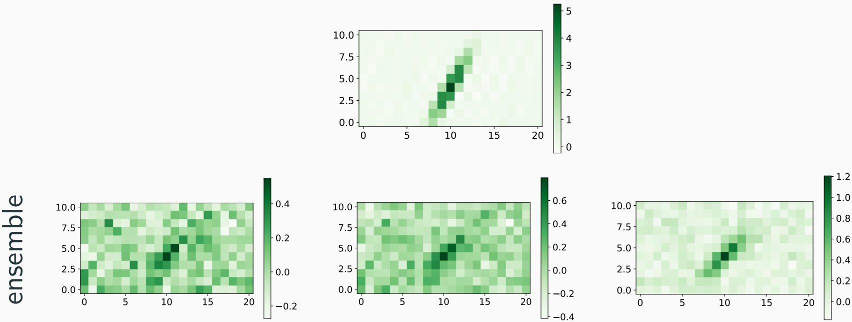


Posteriori (Prod4)

Reduction in data mismatch

A standard iterative ensemble smoother is unable to assimilate the data for the 2D hierarchical model.

One reason for the failure of standard IES



$N_e = 100$ $N_e = 200$ $N_e = 800$

Comparison of purely ensemble based estimates of the sensitivity of observation 27 to the values of z (top row) to the correct value for realization 1 (bottom row). Note the difference in orientation.

Data assimilation – hybrid-IES

Proposing a hybrid IES approach,

$$\delta x_{\ell+1} = -(x_\ell - x^*) - C_x G_\ell^{\mathrm{T}} \left(C_d + G_\ell C_x G_\ell^{\mathrm{T}} \right)^{-1} \times \left(g(m_\ell) + \epsilon^* - d - G_\ell (x_\ell - x^*) \right)$$

but instead of $G \approx \Delta d_\ell \Delta x_\ell^{-1}$ we use

$$G^{\mathrm{T}} = \nabla_x \big(m^{\mathrm{T}} \big) \cdot \nabla_m \big(g^{\mathrm{T}} \big) = \nabla_x \big(m^{\mathrm{T}} \big) \cdot G_m^{\mathrm{T}} \tag{1}$$

with $G_m \approx \Delta d_\ell \Delta m_\ell^{-1}$.

Data assimilation – hybrid-IES[3]

Substituting $G = G_m M_x$ into the update expression with ensemble representation of G_m results in a hybrid data assimilation approach

$$\begin{aligned}\delta x = &-(x - x^*) \\ &- C_x M_x^{\mathrm{T}} (\Delta m)^{-\mathrm{T}} \Delta d^{\mathrm{T}} \left(C_d + \Delta d \, (\Delta m)^{-1} M_x C_x M_x^{\mathrm{T}} (\Delta m)^{-\mathrm{T}} \Delta d^{\mathrm{T}} \right)^{-1} \\ &\times \left(g(m) + \epsilon^* - d - \Delta d \, (\Delta m)^{-1} M_x (x - x^*) \right)\end{aligned}$$

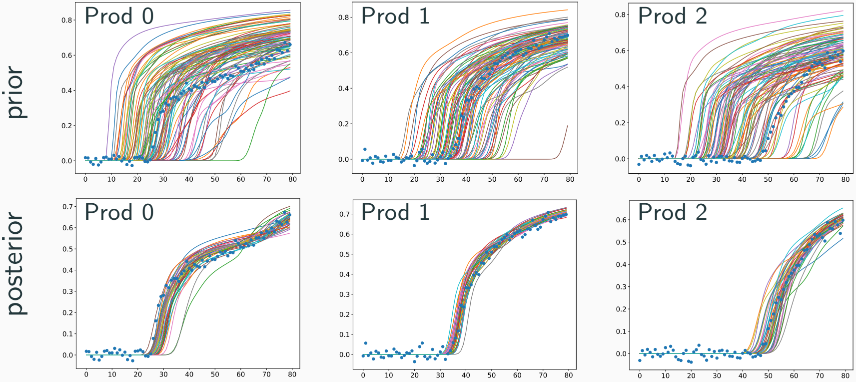
Note: each ensemble member has its own Kalman gain matrix since

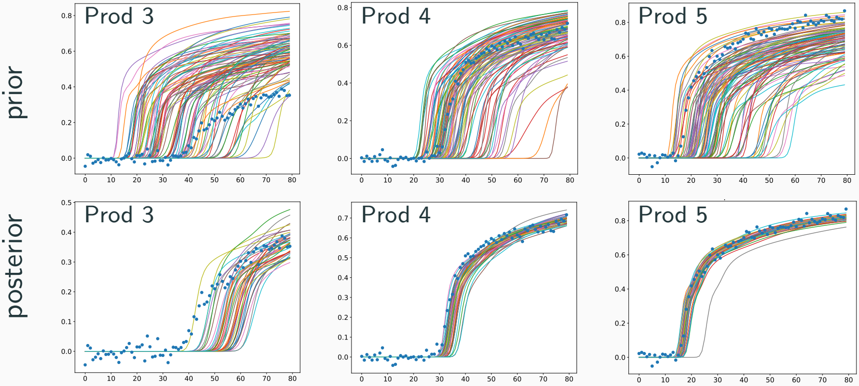
$$M_x = \left[C_m^{1/2} \quad \left(\tfrac{\partial}{\partial \theta} C_m^{1/2}\right) z \right] \tag{2}$$

where θ denotes the hyperparameters.

[3] Oliver (2022)

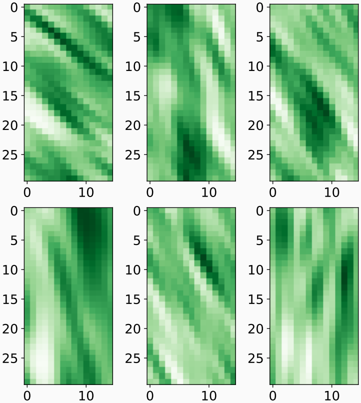
Data assimilation with the hybrid-IES



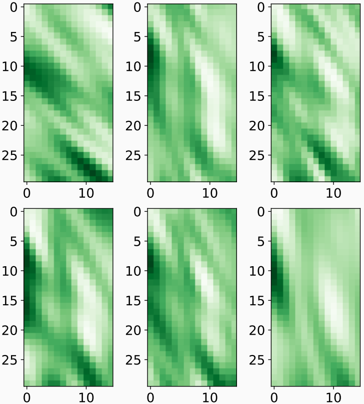


All matched well except Producer 3

Samples from the prior and corresponding samples from the posterior

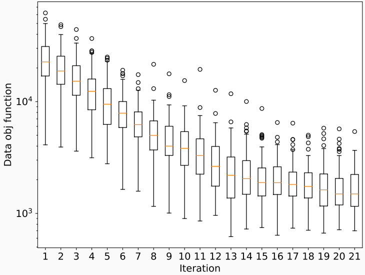


prior



posterior

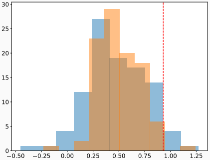
Convergence



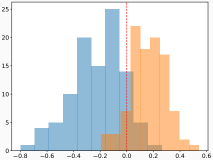
Reduction in squared mismatch with observed (not perturbed) data. Expected value at convergence is 240.

Hyperparameters

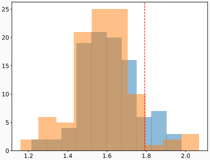
Red dashed line shows true parameter, blue is prior distribution, orange is posterior distribution.



ϕ

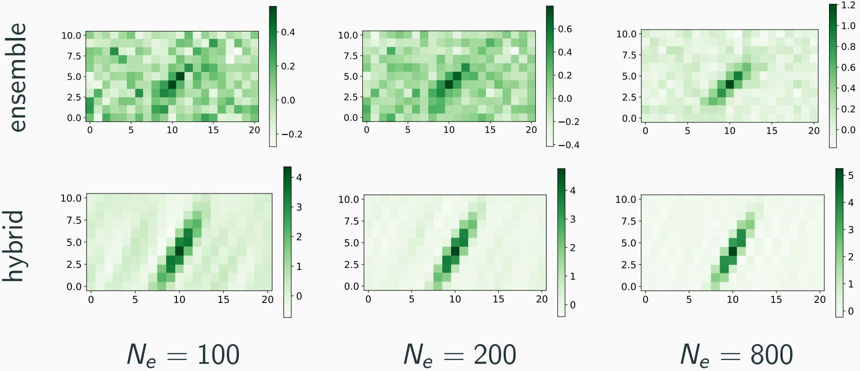


$\ln \rho$



$\ln \alpha$

Short explanation of the reason for failure



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    - Better able to assimilate data
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    - Greater ability to match data than a "bad" prior
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1. Need to develop prior models that are more “forgiving” of model imperfections
    - Better able to assimilate data
    - Reduce need to manually rebuild models
2. Hierarchical models allow for uncertainty in the parameters that characterize the prior
    - Greater ability to match data than a “bad” prior
    - Greater non-linearity and more difficult to history match
    - Iterative ensemble smoothers will often fail
3. Hybrid-IES for hierarchical models
    - Much better at assimilating data
    - Not limited by Gaussian assumptions
    - Currently slow because of cost of computing $C^{1/2}$ for each ensemble member. Consider using $C^{-1/2}$, which may be sparse (Roininen et al., 2019).

References

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