# Graphical Sparse Precision Matrix Estimation and the Ensemble Information Filter

Berent, Feda and Sondre

equinor

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Balestrand, Norway
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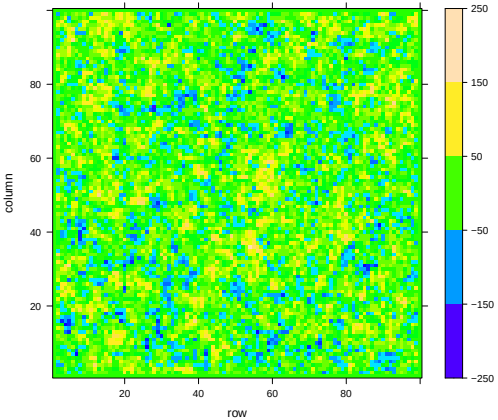
# Who we are: A subset of Scientific COmpUting Team (SCOUT)

# A stochastic spatio-temporal dynamical model

**The stochastic heat equation**

with SPDE

$$\partial_t \boldsymbol{u} = \alpha \nabla^2 \boldsymbol{u} + \sigma d\boldsymbol{W}_t$$



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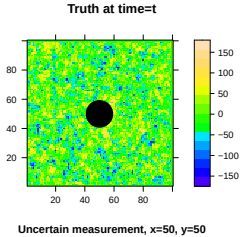

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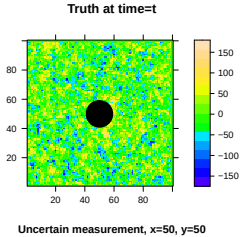


Gaussian prior at time $t-1$

$$\mathcal{N}\left(\boldsymbol{\mu}_{t-1}, \boldsymbol{\Sigma}_{t-1}\right) \underset{100 \text{ realizations}}{\longrightarrow}$$

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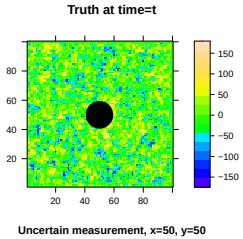


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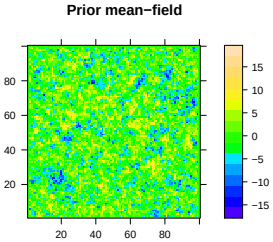

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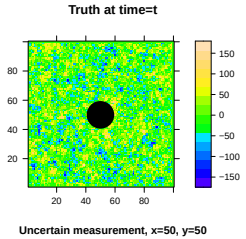


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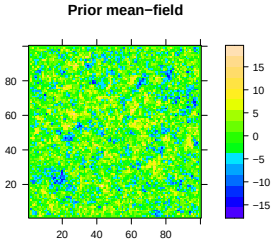


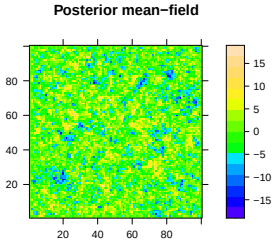


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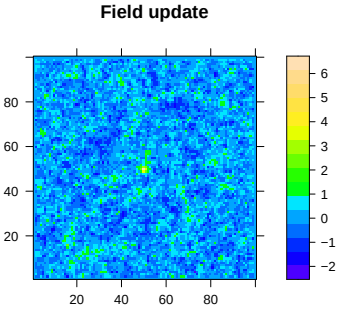

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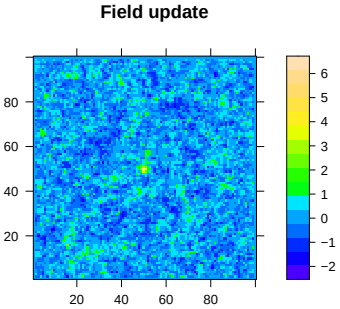


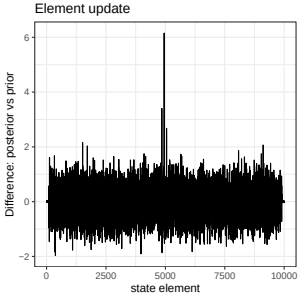

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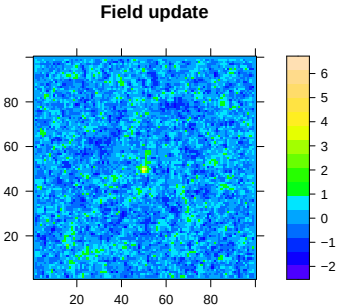


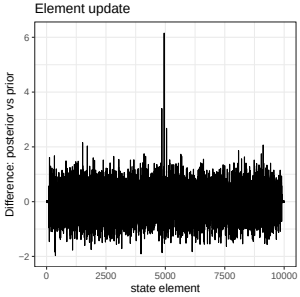


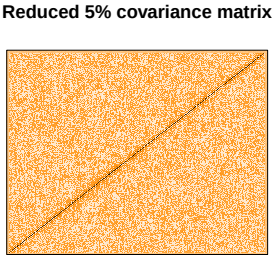

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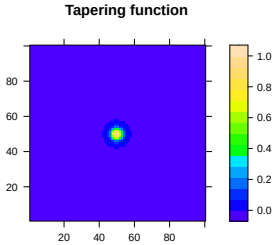
(Anderson, 2003; Evensen, 2003; Ott et al., 2004; Hunt, Kostelich, and Szunyogh, 2007)



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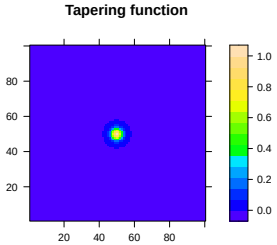


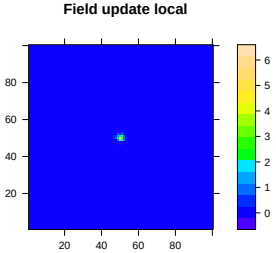


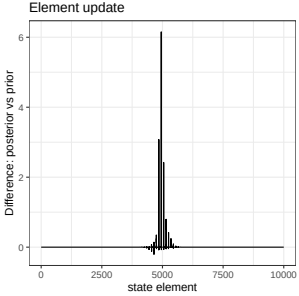

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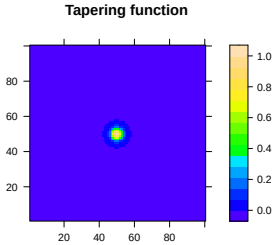


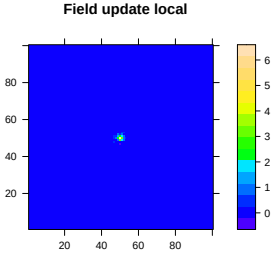


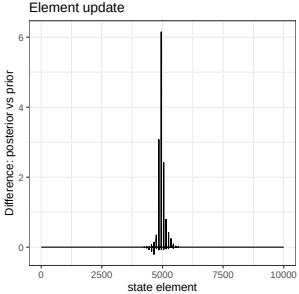

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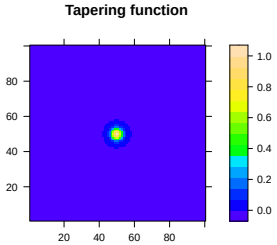
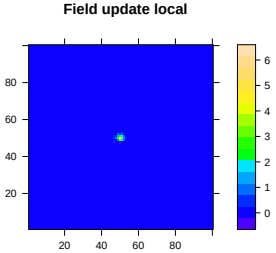
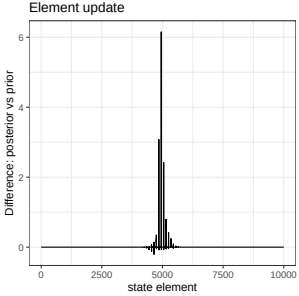
- Involves tuning of hyperparameters (functional form and parameters of tapering function).
- Works on existing (implicit) covariance or residuals - only weakens the *direct* connection between $(i, j)$ and $(k, l)$, but still allow direct connections.



# Reparametrisation: The computational-statistics mindset

**Question**

Is there a way to reparametrise the problem to avoid constrained estimation?

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## Examples:

- Estimate logit-transformed probability instead of constrained probability.

Constrained VS unconstrained

for $0 < p < 1$ do

$$\arg\min_p -\sum_i y_i \log(p) + (1 - y_i)\log(1 - p)$$

VS

given transform $h(\theta) = \dfrac{1}{1 + e^{-\theta}}$ do

$$\arg\min_\theta -\sum_i y_i \log\left(h(\theta)\right) + (1 - y_i)\log\left(1 - h(\theta)\right)$$

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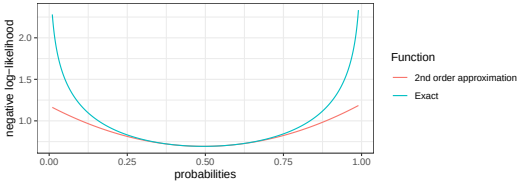

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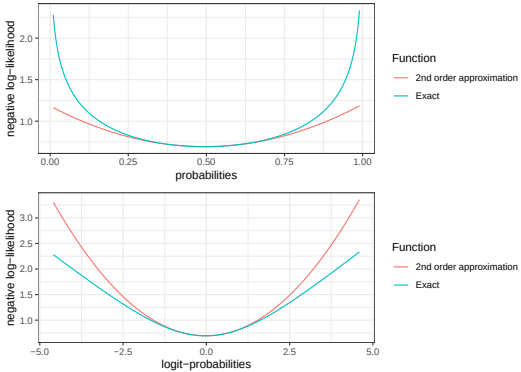

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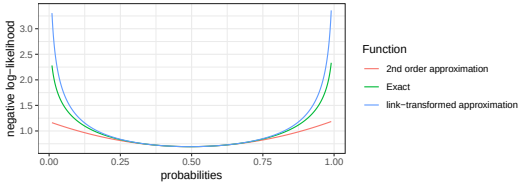


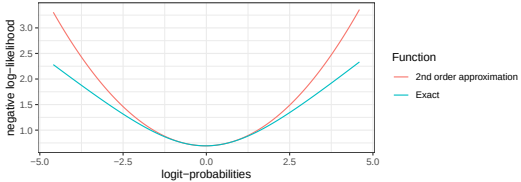

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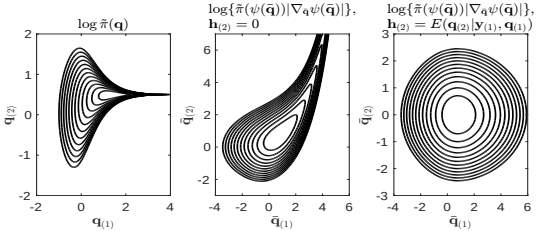
- Constant information parametrisation instead of RMHMC



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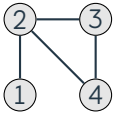
Seek parametrisation with "preference" for local connections/dependence

# Formalising connectivity and locality

**A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$**

- **Vertices** $\mathcal{V} = \{1, \ldots, d\}$
- **Edges** $\mathcal{E} = \{(i,j)\}$ so that $(i,j) \in \mathcal{E}$ if $i$ and $j$ are directly connected
- **Neighbours:** $ne(i) = \{j; (i,j) \in \mathcal{E}\}$



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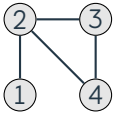


In the example above

- $\mathcal{V} = \{1, 2, 3, 4\}$
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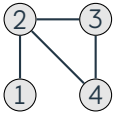


**Markov Random Field (MRF)**

$\boldsymbol{x} \in R^d$ is MRF w.r.t. a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $\mathcal{V} = 1, \ldots, d$ if Markov property

$$x_i \perp x_{-(ne(i),i)} | x_{ne(i)}, \text{ holds } \forall i \in \mathcal{V}$$

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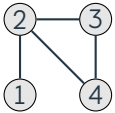


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In the example we would require that

$$x_1 \perp (x_3, x_4) | x_2, \; x_3 \perp x_1 | (x_2, x_4), \; x_4 \perp x_1 | (x_2, x_3)$$

# Markov properties

Keep modelling at a local level...

# Markov properties

## Keep modelling at a local level...

... by using Markov properties

$$E[x_i|x_{-i}] = E[x_i|x_{ne(i)}]$$

and even stronger that

$$x_i|x_{-i} = x_i|x_{ne(i)}$$
$$\Updownarrow$$
$$x_i \perp x_{-(i,ne(i))} = x_i|x_{ne(i)}$$

# Gaussien Markov Random Fields (GMRF)

## Gaussian Markov Random Field (Rue and Held, 2005)

A random vector $\boldsymbol{x} \in R^d$ is a Gaussian Markov Random Field with respect to the graph $\mathcal{G} = (\{1, \ldots, d\}, \mathcal{E})$, with mean $\mu$ and SPD precision matrix $\Lambda$ if

$$p(x) = (2\pi)^{-\frac{d}{2}} \sqrt{|\Lambda|} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^\top \Lambda (\boldsymbol{x} - \boldsymbol{\mu})\right)$$

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$$\Lambda_{i,j} \neq 0 \Leftrightarrow (i,j) \in \mathcal{E} \forall i \neq j.$$

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Notice that connectivity and local dependence is *directly* specified through the non-zero elements of the precision matrix. No other constraints needed.

# GMRF parametrisation matters: Estimation

Connectivity and Markov properties implied by the precision matrix.

- Unconstrained optimisation (ML-estimation)
- Massively important for estimation: Spurious correlations disappear.

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 & \cdots & & & & & \cdots & 0 \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} & 0 & \cdots & & & & \cdots & 0 \\ 0 & \Lambda_{32} & \Lambda_{33} & \Lambda_{34} & 0 & \cdots & & & \cdots & 0 \\ \vdots & & & & & \ddots & & & & \vdots \\ 0 & \cdots & & & \cdots & 0 & \Lambda_{d-2,d-3} & \Lambda_{d-2,d-2} & \Lambda_{d-2,d-18} & 0 \\ 0 & \cdots & & & & \cdots & 0 & \Lambda_{d-1,d-2} & \Lambda_{d-1,d-1} & \Lambda_{d-1,d} \\ 0 & \cdots & & & & & \cdots & 0 & \Lambda_{d,d-1} & \Lambda_{d,d} \end{bmatrix}$$

# GMRF parametrisation matters: Inversion

$$\Sigma = \begin{bmatrix} \Sigma_{AA} & \Sigma_{AB} \\ \Sigma_{BA} & \Sigma_{BB} \end{bmatrix}, \quad \Lambda = \Sigma^{-1}$$

## Covariance parametrisation

## Precision parametrisation

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## Covariance parametrisation

Convenient for working with marginal distributions of $\boldsymbol{x}$

- $E\left[\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\right] = \boldsymbol{A}\boldsymbol{\mu} + \boldsymbol{b}$
- $Var\left[\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\right] = \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^{\top}$

Not for conditional distributions (requires matrix inversion, e.g. Kalman filter).

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## Precision parametrisation

Convenient for working with conditional distributions of $\boldsymbol{x}$

- $E[x_i|x_{-i}] = \mu_i - \frac{1}{\Lambda_{i,i}} \sum_{j \neq i} \Lambda_{i,j}(x_j - \mu_j)$
- $Prec(\boldsymbol{x}_A|\boldsymbol{x}_{-A}) = \Lambda_{A,A}$
- $Corr(x_i, x_j|x_{-ij}) = -\frac{\Lambda_{i,j}}{\sqrt{\Lambda_{i,i}\Lambda j,j}} \; i \neq j$

# Autoregressive (1) example


$$x_t = \phi x_{t-1} + \epsilon_t,\ x_1 \sim \mathcal{N}\left(0, \frac{1}{1-\phi^2}\right),\ \epsilon_t \sim \mathcal{N}(0,1)$$

Covariance parametrisation...

Precision parametrisation...

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... is dense!

$$\mathbf{\Sigma} = \begin{bmatrix} B(1,1) & \cdots & B(1,T) \\ \vdots & \ddots & \vdots \\ B(T,1) & \cdots & B(T,T) \end{bmatrix},\ B(i,j) = \frac{\phi^{|i-j|}}{1-\phi^2}$$

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## Precision parametrisation...

... is sparse!

$$\Lambda = \begin{bmatrix} 1 & -\phi & & & \\ -\phi & 1+\phi^2 & -\phi & & \\ & \ddots & \ddots & \ddots & \\ & & -\phi & 1+\phi^2 & -\phi \\ & & & -\phi & 1 \end{bmatrix}$$

# The Information Filter (Moore and Anderson, 1979)

Employs the canonical parametrization of the multivatiate Gaussian:

$$\boldsymbol{\nu} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}, \boldsymbol{\Lambda} = \boldsymbol{\Sigma}^{-1}$$

**Predict step**

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# Extension to the ensemble variant?

**Sample from belief**

$$\boldsymbol{x}_{t-1|t-1}^{(i)} \sim \mathcal{N}\left(\boldsymbol{\mu}_{t-1|t-1}, \boldsymbol{\Lambda}_{t-1|t-1}\right) \; i = 1, \ldots, n$$

**Predict**

$$\boldsymbol{x}_{t|t-1}^{(i)} = g(\boldsymbol{x}_{t-1|t-1}^{(i)})$$

**Estimate**

Using sample $\{\boldsymbol{x}_{t|t-1}^{(i)}\}_{i=1}^{n}$ estimate $\hat{\boldsymbol{\mu}}_{t|t-1}$ and $\hat{\boldsymbol{\Lambda}}_{t|t-1}$ w.r.t. graph $\mathcal{G}$

**Update**

$$\hat{\boldsymbol{\nu}}_{t|t-1} = \hat{\boldsymbol{\Lambda}}_{t|t-1}\hat{\boldsymbol{\mu}}_{t|t-1}$$
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- Extension of the information filter to an ensemble variant is not straight forward.

Background

Graphical Sparse Precision Matrix Estimation

The Ensemble Information Filter

Examples

# The sample covariance estimate...

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{n-1} \sum_i (\boldsymbol{x}_i - \bar{\boldsymbol{x}})(\boldsymbol{x}_i - \bar{\boldsymbol{x}})^\top$$

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# (Stein-type) shrinkage of the covariance estimator

**We employ the Ledoit-Wolf shrinkage estimator** (Ledoit and Wolf, 2004)

$$\hat{\boldsymbol{\Sigma}}_{\boldsymbol{T}} = (1-\lambda)\hat{\boldsymbol{\Sigma}} + \lambda \boldsymbol{T}$$

Targets to fix stability, singularity and efficiency.

## In general...

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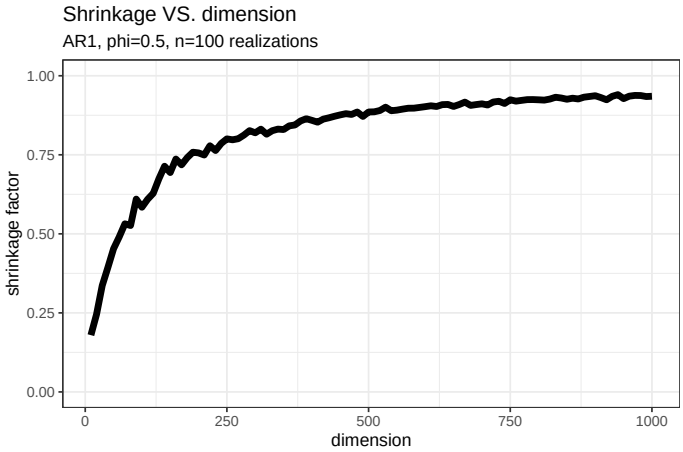
- The SCV diagonal $\boldsymbol{T} = \mathtt{diag}\left(\hat{\boldsymbol{\Sigma}}\right)$
- Asymptotic optimality results for shrinkage factor $\hat{\lambda}$ (Touloumis, 2015)

# The Stein-type covariance shrinkage estimate: In high dimensions



# Asymptotic Stein-type shrinkage

**Very smart!**

1. Non-parametric model with loose moment conditions
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**Lacks the informed structure of locality $\rightarrow$ shrinks off-diagonal elements to zero**

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2017 Tuning Intensive Graph Estimation and Regression (TIGER and EPIC) (Zhao and Liu, 2014; Liu and Wang, 2017)

# Precision matrix estimation with respect to a graph

The algorithms estimate the graph through Lasso-type algorithms (L1-penalization)

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- **Non-parametric precision matrix estimation with respect to a known graphical structure**

# SPD precision matrix estimation with respect to a graph

## The method of Le and Zhong (2022)

- Column-by-column sub-sample covariance estimate inversion
- Sub-sample covariance estimate blocks identified due to the knowledge of the graph $\mathcal{G}$

## Pitfalls

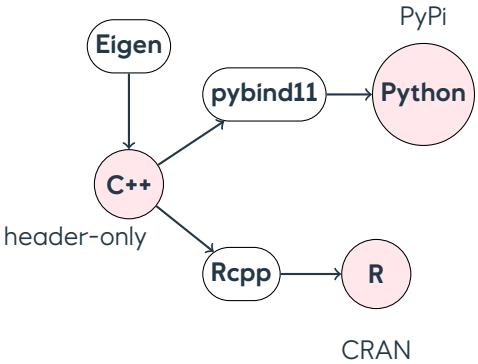
- Not necessarily positive definite
- Not necessarily symmetric

## Add some ingredients

- The efficient asymptotic shrinkage of Touloumis (2015)
- Symmetry conversion: $\hat{\mathbf{\Lambda}} = \frac{1}{2}\left(\tilde{\mathbf{\Lambda}} + \tilde{\mathbf{\Lambda}}^{\top}\right)$

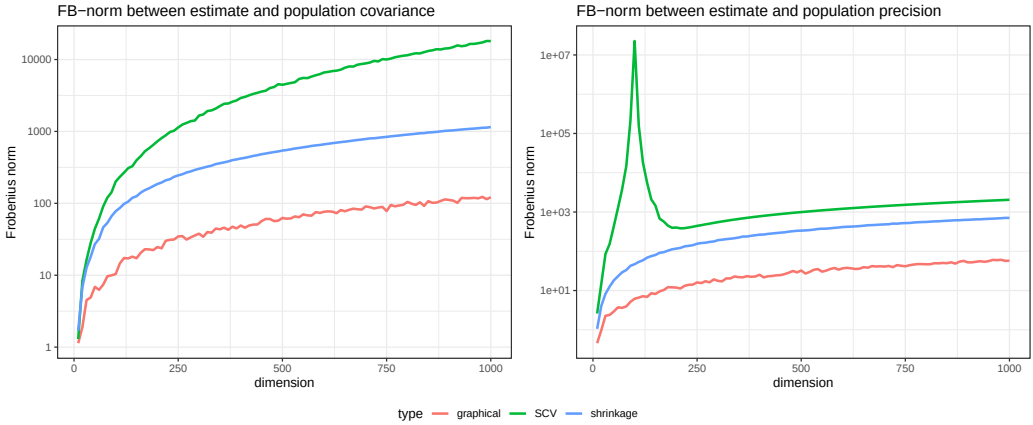
# GraphSPME: Graphical Sparse Precision Matrix Estimation

**Open sourced at** `https://github.com/equinor/GraphSPME`
**Paper at** `https://arxiv.org/abs/2205.07584`




```python
from graphspme import prec_sparse, cov_shrink_spd
import numpy as np
from scipy import sparse
def rar1(T, phi):
    ...
# Sample data
n, p, psi = 100, 100, 0.6
x = np.tile(rar1(p,psi), (n,1))
# Estimate covariance
Sigma = cov_shrink_spd(x)
# Estimate precision
diagonals = [[1]*p,[1]*(p-1),[1]*(p-1)]
G = sparse.diags(diagonals, [0, -1, 1], format="csr")
Prec = prec_sparse(x, G)
```

# AR1 estimation results



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- A typical remedy is to use shrinkage, giving SPD estimates. But does not utilize information of sparsity
- GraphSPME combines the method of Le and Zhong (2022) for precision estimation w.r.t. a graph, with asymptotic shrinkage methods Touloumis (2015) to ensure SPD estimates.

Background

Graphical Sparse Precision Matrix Estimation

The Ensemble Information Filter

Examples

# The Ensemble Information Filter

**Sample from belief**

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**Predict**

$$\boldsymbol{x}_{t|t-1}^{(i)} = g(\boldsymbol{x}_{t-1|t-1}^{(i)})$$

**Estimate**

Using sample $\{\boldsymbol{x}_{t|t-1}^{(i)}\}_{i=1}^{n}$ estimate $\hat{\boldsymbol{\mu}}_{t|t-1}$ and $\hat{\boldsymbol{\Lambda}}_{t|t-1}$ w.r.t. graph $\mathcal{G}$ (using GraphSPME)

**Update**

$$\hat{\boldsymbol{\nu}}_{t|t-1} = \hat{\boldsymbol{\Lambda}}_{t|t-1}\hat{\boldsymbol{\mu}}_{t|t-1}$$
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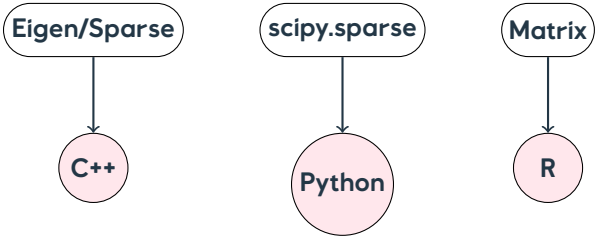
- `sparse_prec` efficiently estimates a sparse precision matrix according to the GraphSPME algorithm.
- $\hat{\boldsymbol{\Lambda}}$ is guaranteed to be symmetric positive definite by the GraphSPME algorithm, and a solution to the mean-precision parametrisation thus always exists, furthermore the covariance matrix may in principle be retrieved.

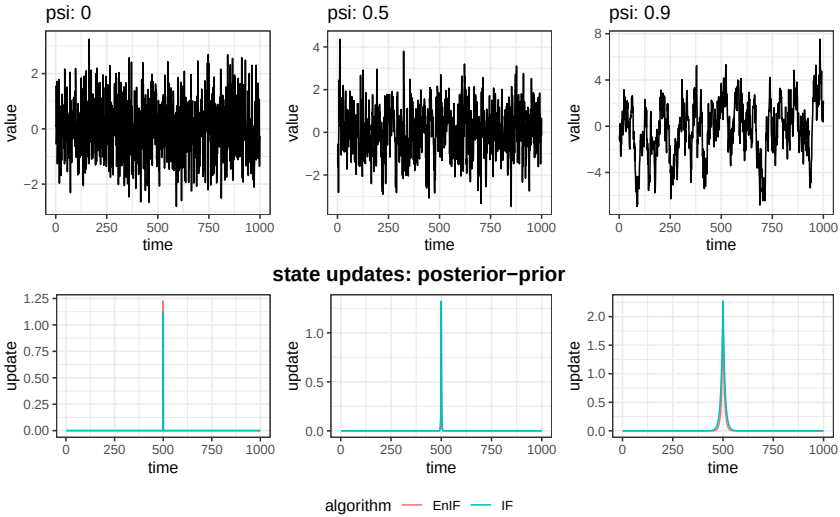
# Implementation

**Examples at** `https://github.com/equinor/Enkf-Workshop-2022`

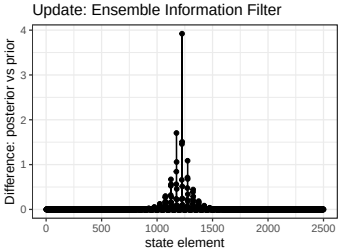
Generally easy to implement, but requires a sparse matrix library with corresponding sparse linear solvers.

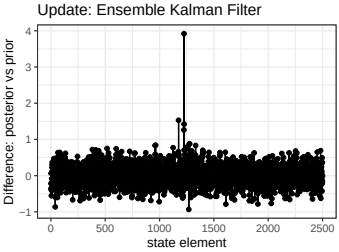


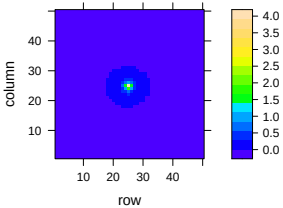
# AR1 updates

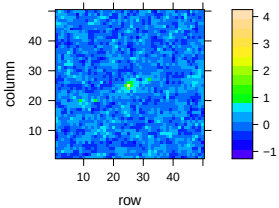

psi: 0
psi: 0.5
psi: 0.9
value
time
state updates: posterior−prior
update
algorithm
EnIF
IF

# Random field updates, $n = 200$ vs $n = 10000$



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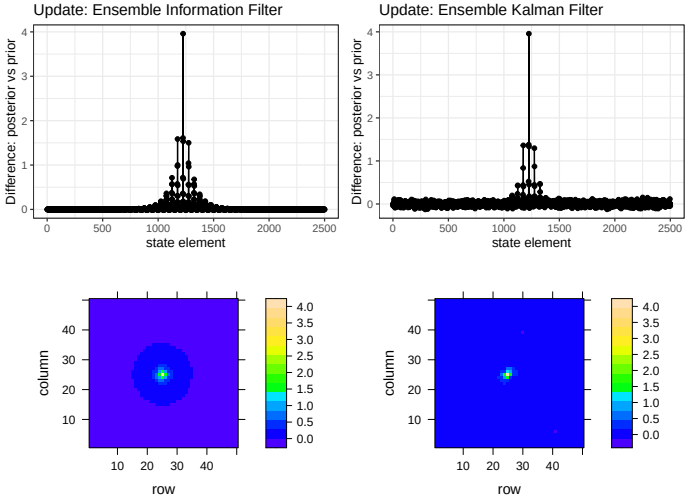

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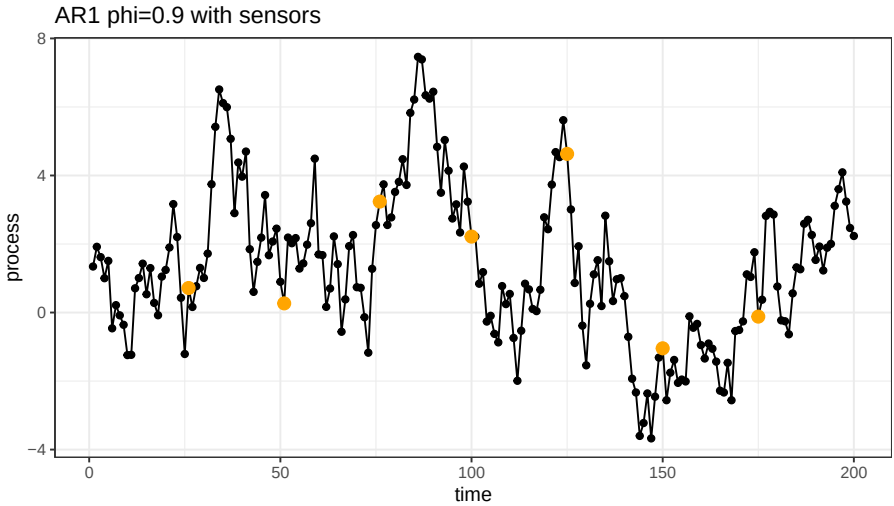
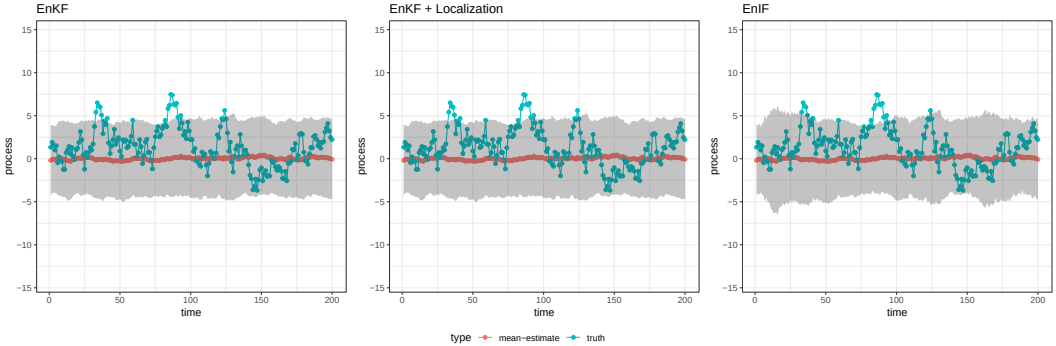
- GraphSPME allows the extension to the Ensemble Information Filter.
- The numerical linear algebra relies heavily on smart algorithms for sparse matrices and linear solvers.
- Filtering updates are much less noisy than for that of the EnKF, and seems to solve the problem of localisation without requiring tuning of arbitrary kernels.

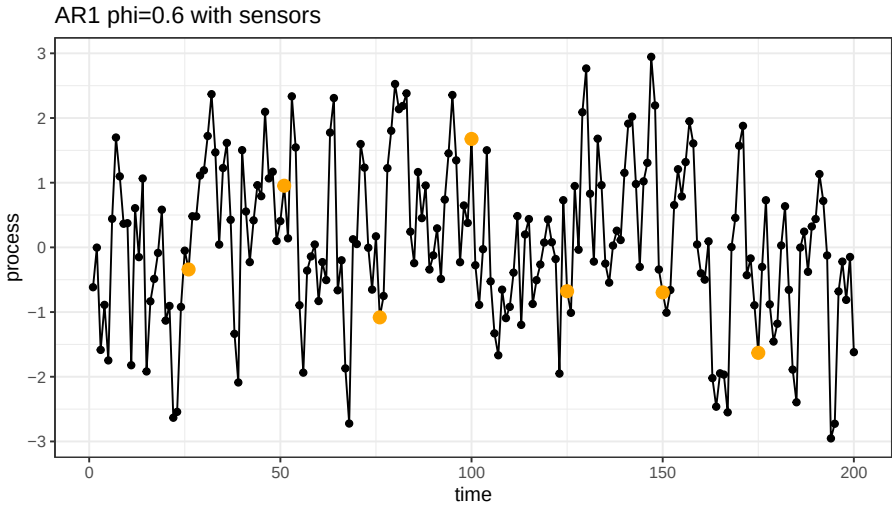
# AR1, strong dependence



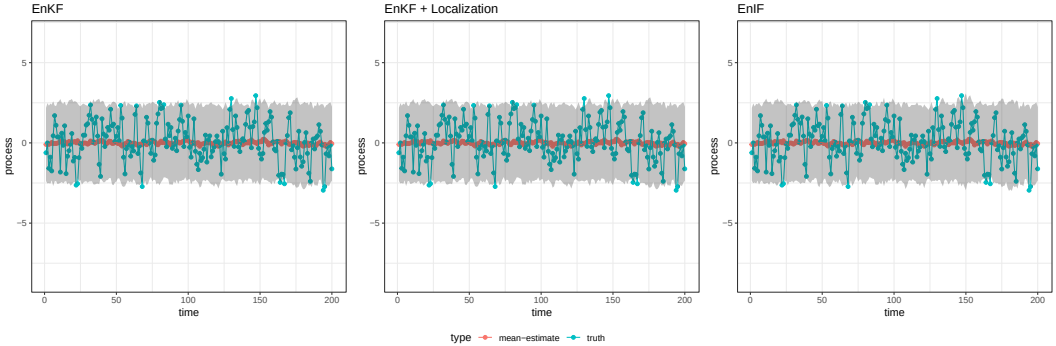
# AR1: $\phi = 0.9$


EnKF
EnKF + Localization
EnIF
process
time
15
10
5
0
−5
−10
−15
0
50
100
150
200
type
mean−estimate
truth

# AR1, middle dependence



# AR1: $\phi = 0.6$


EnKF
EnKF + Localization
EnIF
process
time
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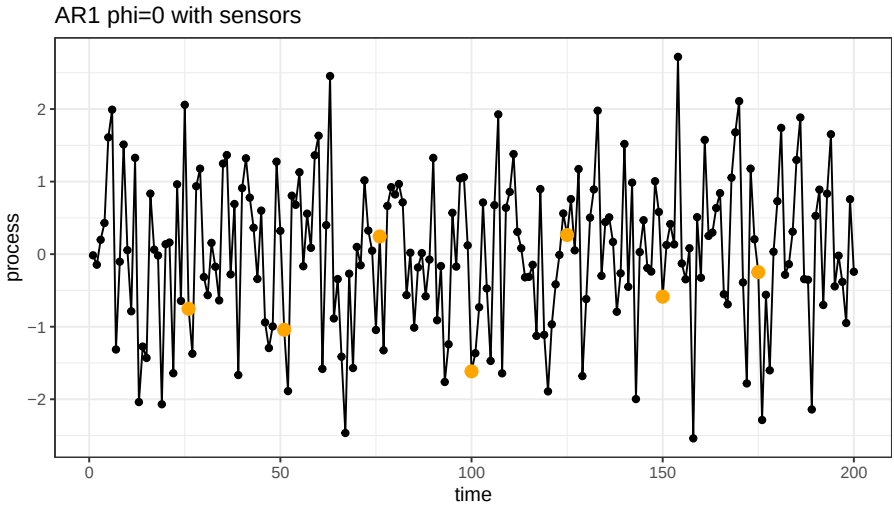
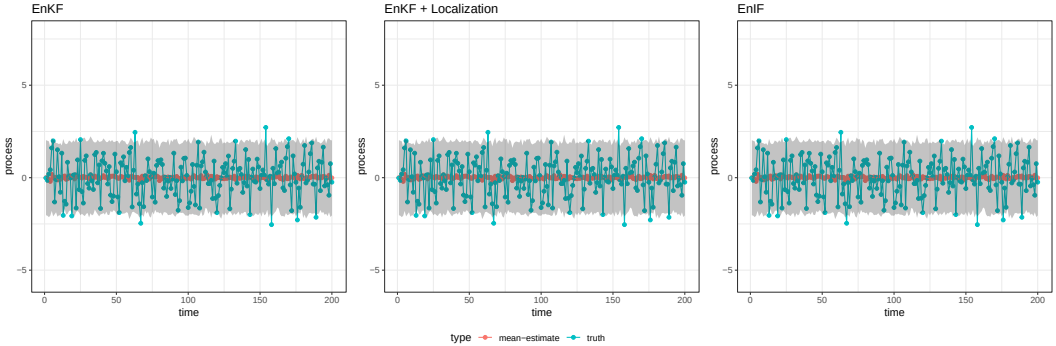
type
mean−estimate
truth

# AR1, no dependence, noise



# AR1: $\phi = 0.0$


EnKF
EnKF + Localization
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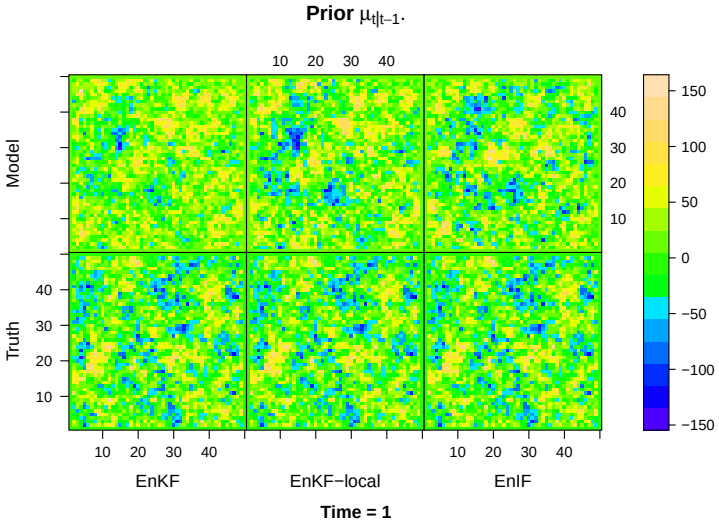
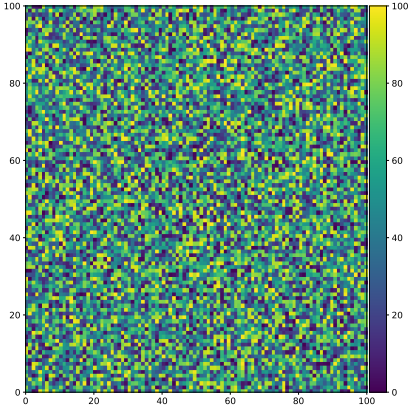
truth

# Stochastic heat equation: Pure state estimation



# Stochastic heat equation: Combined parameter-state estimation

$\partial_t \boldsymbol{u} = \alpha \nabla^2 \boldsymbol{u} + \sigma d\boldsymbol{W}_t,\ (\alpha, \sigma)$ are unknown and must be estimated.



# Stochastic heat equation: Combined parameter-state estimation

$$\partial_t \boldsymbol{u} = \alpha \nabla^2 \boldsymbol{u} + \sigma d\boldsymbol{W}_t, \ (\alpha, \sigma) \text{ are unknown and must be estimated.}$$

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- Have not yet started to think about the joint estimation problem.
- In principle, possible to do in the same way as for EnKF (assume additive and specify a fully connected graph).
- Perhaps more elegant with the two-step estimation procedure e.g. when using the Laplace approximatin to integrate out a latent state.

# Full recap

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# Full recap

- Filtering algorithms needs to be informed on locality to avoid noisy updates due to $n << p$.
- A natural assumption for spatio-temporal models is assumptions of Markov properties.
- For Gaussian Markov Random Fields, the precision matrix is sparse. The corresponding information filter utilizes this.
- We allow the extension to the Ensemble Information Filter by creating GraphSPME: graphical sparse precision matrix estimation
- The filtering updates from the ensemble information filter seem to be smooth and comparatively without noise (information efficient). The method is free of tuning and arbitrary measures of distance.

# Questions?

# Bibliography I


- Anderson, Jeffrey L (2003). "A local least squares framework for ensemble filtering". In: *Monthly Weather Review* 131.4, pp. 634–642.
- Burgers, Gerrit, Peter Jan Van Leeuwen, and Geir Evensen (1998). "Analysis scheme in the ensemble Kalman filter". In: *Monthly weather review* 126.6, pp. 1719–1724.
- Cai, Tony, Weidong Liu, and Xi Luo (2011). "A constrained â□□ 1 minimization approach to sparse precision matrix estimation". In: *Journal of the American Statistical Association* 106.494, pp. 594–607.
- Evensen, Geir (1994). "Sequential data assimilation with a nonlinear quasi-geostrophic model using Monte Carlo methods to forecast error statistics". In: *Journal of Geophysical Research: Oceans* 99.C5, pp. 10143–10162.
- – (2003). "The ensemble Kalman filter: Theoretical formulation and practical implementation". In: *Ocean dynamics* 53.4, pp. 343–367.

# Bibliography II


- Friedman, Jerome, Trevor Hastie, and Robert Tibshirani (2008). "Sparse inverse covariance estimation with the graphical lasso". In: *Biostatistics* 9.3, pp. 432–441.
- Hastie, Trevor et al. (2009). *The elements of statistical learning: data mining, inference, and prediction*. Vol. 2. Springer.
- Hunt, Brian R, Eric J Kostelich, and Istvan Szunyogh (2007). "Efficient data assimilation for spatiotemporal chaos: A local ensemble transform Kalman filter". In: *Physica D: Nonlinear Phenomena* 230.1-2, pp. 112–126.
- Le, Thien-Minh and Ping-Shou Zhong (2022). "High-dimensional precision matrix estimation with a known graphical structure". In: *Stat* 11.1, e424.
- Ledoit, Olivier and Michael Wolf (2004). "A well-conditioned estimator for large-dimensional covariance matrices". In: *Journal of multivariate analysis* 88.2, pp. 365–411.

# Bibliography III


- Liu, Han and Lie Wang (2017). "Tiger: A tuning-insensitive approach for optimally estimating gaussian graphical models". In: *Electronic Journal of Statistics* 11.1, pp. 241–294.
- Moore, John Barratt and B Anderson (1979). *Optimal filtering*. Prentice-Hall New York.
- Nino-Ruiz, Elias D, Luis Guzman, and Daladier Jabba (2021). "An ensemble kalman filter implementation based on the ledoit and wolf covariance matrix estimator". In: *Journal of Computational and Applied Mathematics* 384, p. 113163.
- Ott, Edward et al. (2004). "A local ensemble Kalman filter for atmospheric data assimilation". In: *Tellus A: Dynamic Meteorology and Oceanography* 56.5, pp. 415–428.
- Rue, Havard and Leonhard Held (2005). *Gaussian Markov random fields: theory and applications*. Chapman and Hall/CRC.

# Bibliography IV


- Touloumis, Anestis (2015). "Nonparametric Stein-type shrinkage covariance matrix estimators in high-dimensional settings". In: *Computational Statistics & Data Analysis* 83, pp. 251–261.
- Yuan, Ming (2010). "High dimensional inverse covariance matrix estimation via linear programming". In: *The Journal of Machine Learning Research* 11, pp. 2261–2286.
- Zhao, Tuo and Han Liu (2014). "Calibrated precision matrix estimation for high-dimensional elliptical distributions". In: *IEEE transactions on Information Theory* 60.12, pp. 7874–7887.
- Zhou, Shuheng et al. (2011). "High-dimensional covariance estimation based on Gaussian graphical models". In: *The Journal of Machine Learning Research* 12, pp. 2975–3026.