Parametric Kalman filter : toward an alternative to the EnKF?

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Kalman Filter

[Kalman, 1960] filter details the dynamics of Gaussian uncertainty along the analysis and linear forecast cycles.
Analysis update writes

$$\left\{\begin{array}{l}\mathbf{K}=\mathbf{P}^f\mathbf{H}^T(\mathbf{H}\mathbf{P}^f\mathbf{H}^T+\mathbf{R})^{-1},\\ \mathcal{X}^a=\mathcal{X}^f+\mathbf{K}(\mathcal{Y}^o-\mathbf{H}\mathcal{X}^f),\\ \mathbf{P}^a=(\mathbf{I}-\mathbf{K}\mathbf{H})\mathbf{P}^f,\end{array}\right. \tag{1}$$

where $\mathbf{P}^f=\mathbb{E}[\varepsilon^f{\varepsilon^f}^T]$ and $\mathbf{P}^a=\mathbb{E}[\varepsilon^a{\varepsilon^a}^T]$, with the forecast evolution

$$\left\{\begin{array}{l}\mathcal{X}^f=\mathbf{M}\mathcal{X}^a,\\ \mathbf{P}^f=\mathbf{M}\mathbf{P}^a\mathbf{M}^T.\end{array}\right. \tag{2}$$

This is a simple algorithm. But update of forecast covariance matrix $\mathbf{P}^f=\mathbf{M}\mathbf{P}^a\mathbf{M}^T$ is numerically costly.

Ensemble Kalman Filter

Extending to the non-linear setting, EnKF estimates covariance matrices from samples [Evensen, 1994]

$$\mathbf{P}^f{}_e = \frac{1}{N-1}\sum_k \varepsilon^f{}_k {\varepsilon^f{}_k}^T, \tag{3}$$

where $\varepsilon^f{}_k = \mathcal{X}_k - \overline{\mathcal{X}_k}$, with the update equation

$$\mathcal{X}^a{}_k = \mathcal{X}^f{}_k + \mathbf{K}_e(\mathcal{Y}^o + \varepsilon^o{}_k - \mathbf{H}\mathcal{X}^f{}_k). \tag{4}$$

- EnKF is a robust algorithm, it runs for Lorenz 63 as well as for ocean/atmosphere,
- But it is suffering from sampling noise (localization, imperfect balances)
- Parallel implementation is natural,
- But often consists in computation of same model, at low resolution.

Ensemble Kalman Filter

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Is it possible to describe covariance evolution at low cost, without ensemble, while resolving KF equations ?

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Principle of Parametric Kalman Filter

Idea: replace the ensemble by anisotropic covariance model

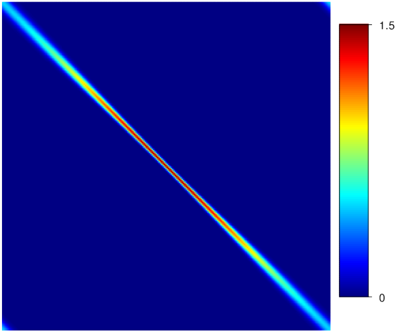
Parametric Kalman Filter

1. Consider a parametric covariance model,
2. Write parameter dynamics along analysis and forecast cycles.

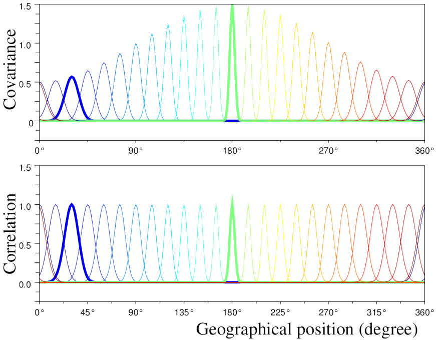
Example of interesting parameters:

the variance and the length-scale

Covariance matrix: Variance and Length-scale fields

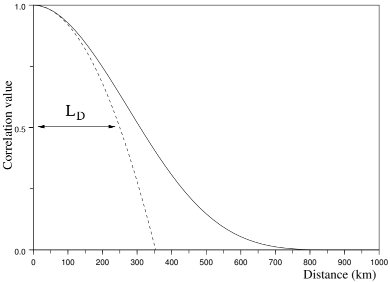


Covariance matrix



Covariance/Correlation functions

The correlation length-scale

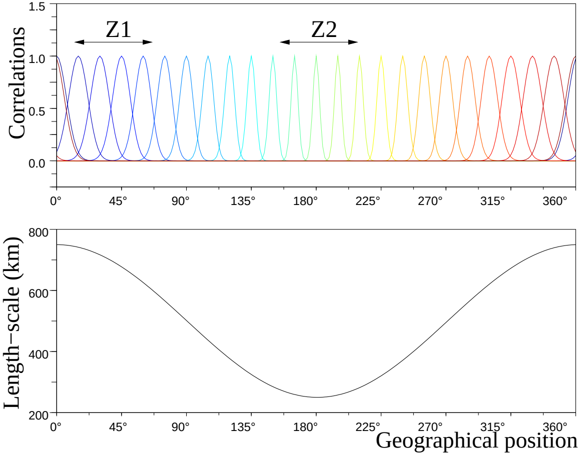


One correlation function with its second order Taylor's approximation.

$$\rho(x, x+\delta x) = 1 - \frac{1}{2}\frac{\delta x^2}{L_D^2} + o(\delta x^2)$$

L_D is the the correlation length-scale

Example of length-scale field for the 1D example:



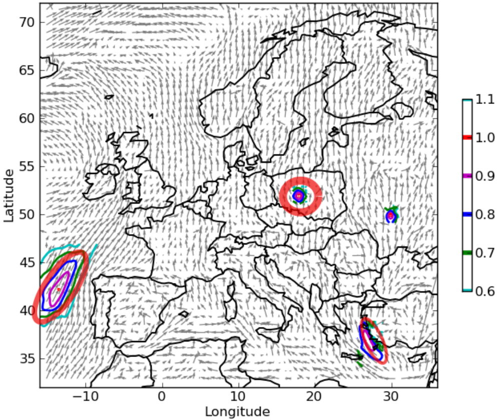
Extension of the length-scale in 2D/3D: the local metric tensor

In 2D/3D, Taylor's expansion writes

$$\rho(\mathbf{x},\mathbf{x}+\delta\mathbf{x}) = 1 - \frac{1}{2}||\delta\mathbf{x}||^2_{\boldsymbol{g}_\mathbf{x}} + \mathcal{O}(||\delta\mathbf{x}||^3),\ \text{with}\ [\boldsymbol{g}_\mathbf{x}]_{ij} = -\partial^2_{ij}\rho, \quad (5)$$

where the local metric $\boldsymbol{g}_\mathbf{x}$ features the shape of the correlation function in the vicinity of point **x**.

Diagnosis of the anisotropy field, comparison with mean flow



Mean flow and Anisotropy for few correlation functions [Jaumouillé et al., 2013]

PKF: diffusion based covariance model

In the error covariance model based on the diffusion equation [Weaver and Courtier, 2001],

$$\mathbf{P} = \Sigma \mathbf{C} \Sigma, \tag{6}$$

where Σ stands for the diagonal matrix of grid-points standard-deviation and

$$\mathbf{C} = \mathbf{L}\mathbf{L}^T, \tag{7}$$

is the correlation matrix with $\mathbf{L} = e^{\mathcal{L}\frac{1}{2}}$, $e^{\mathcal{L}t}$ beeing the propagateur of the diffusion equation

$$\partial_t u = \mathcal{L}(u) = \nabla \cdot (\boldsymbol{\nu} \nabla u), \tag{8}$$

Following [Pannekoucke and Massart, 2008], the local diffusion tensor field is set by

$$\boldsymbol{\nu}_{\mathbf{x}} = \frac{1}{2} \boldsymbol{g}_{\mathbf{x}}^{-1} \tag{9}$$

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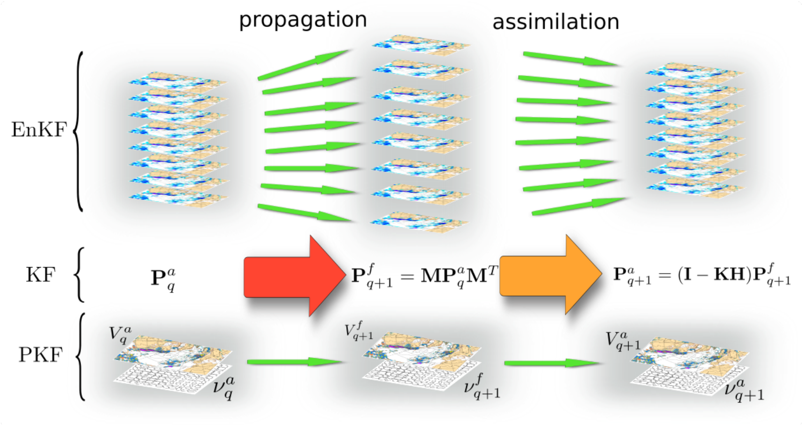
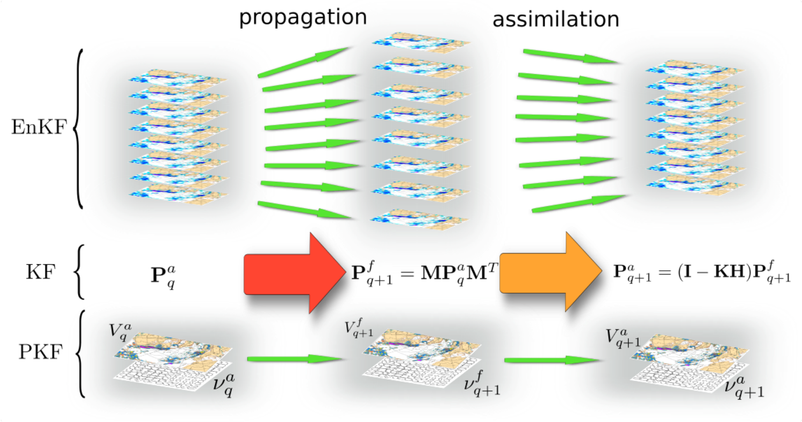
Variance and metric fields are the parameters of the covariance model based on the diffusion equation.

Principle of Parametric Kalman Filter



Principle of Parametric Kalman Filter



What are the PKF equations for the analysis and forecast steps ?

Analysis update of the parametric formulation

Require: Fields of $\boldsymbol{\nu}^b$ and $\boldsymbol{V}^b$, $\boldsymbol{V}^o$ and location $\boldsymbol{x}_j$ of the p observations to assimilate

for $j = 1 : p$ **do**

  0- Initialization of intermediate quantities

  $V_j^b = V_{\boldsymbol{x}_j}^b,\ V_j^o = V_{\boldsymbol{x}_j}^o,\ \boldsymbol{\nu}_j = \boldsymbol{\nu}_{\boldsymbol{x}_j}^b$

  $\rho_j(\boldsymbol{x}) = \exp\left(-\frac{1}{4}||\boldsymbol{x} - \boldsymbol{x}_j||^2_{\boldsymbol{\nu}_j^{-1}}\right)$

  1- Computation of analysis statistics

  $V_{\boldsymbol{x}}^a = V_{\boldsymbol{x}}^b\left(1 - \rho_j^2(\boldsymbol{x})\frac{V_j^b}{V_j^b + V_j^o}\right)$

  $\boldsymbol{\nu}_{\boldsymbol{x}}^a = \boldsymbol{\nu}_{\boldsymbol{x}}^b\left(1 - \rho_j^2(\boldsymbol{x})\frac{V_j^b}{V_j^b + V_j^o}\right)$

  2- Update of the background statistics

  $V_{\boldsymbol{x}}^b \leftarrow V_{\boldsymbol{x}}^a$

  $\boldsymbol{\nu}_{\boldsymbol{x}}^b \leftarrow \boldsymbol{\nu}_{\boldsymbol{x}}^a$

end for

Return fields $\boldsymbol{\nu}^a$ and $\boldsymbol{V}^a$

Algorithm 1: Iterated process building analysis covariance matrix at the leading order, under Gaussian shape assumption. [Pannekoucke et al., 2016]

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Parameter dynamics for linear advection/diffusion

For the particular linear advection-diffusion dynamics

$$\partial_t \alpha + u \partial_x \alpha = \kappa \partial_x^2 \alpha, \tag{14}$$

where u denotes the velocity and κ the diffusion rate, the time evolution of the variance and the diffusion tensor is given at the lead order by

$$\left\{ \begin{array}{l} \partial_t \boldsymbol{\nu}^f + \boldsymbol{u} \nabla \boldsymbol{\nu}^f = \boldsymbol{\nu}^f (\nabla \boldsymbol{u})^T + (\nabla \boldsymbol{u}) \boldsymbol{\nu}^f + 2\boldsymbol{\kappa}, \\ \partial_t \boldsymbol{V}^f + \boldsymbol{u} \nabla \boldsymbol{V}^f = -\boldsymbol{V}^f Tr\left[\left(\boldsymbol{\nu}^f\right)^{-1} \boldsymbol{\kappa} \right]. \end{array} \right. \tag{15}$$

There is a coupling between the error variance and local diffusion tensor fields due to the diffusion process.

Simple analysis/forecast cycle

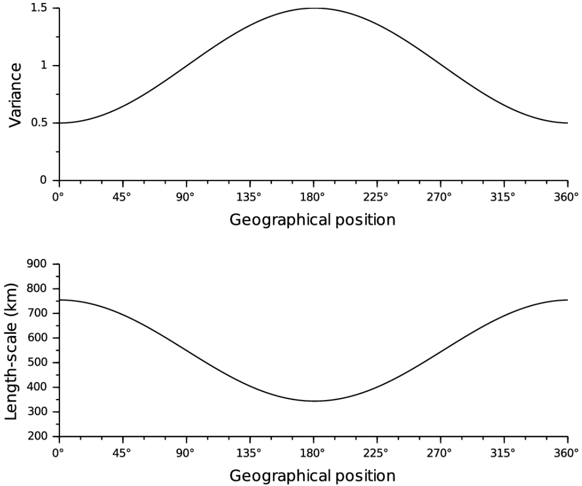


Figure: -1- Initial condition for the background error variance field (top) and length-scale field $L_x = \sqrt{2\nu_x}$ (bottom).

Simple analysis/forecast cycle

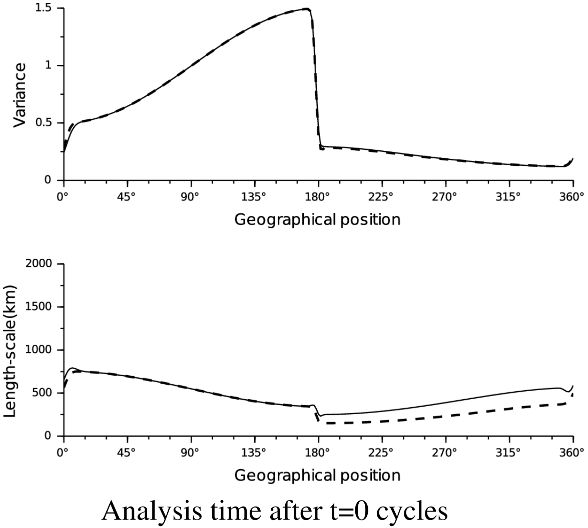
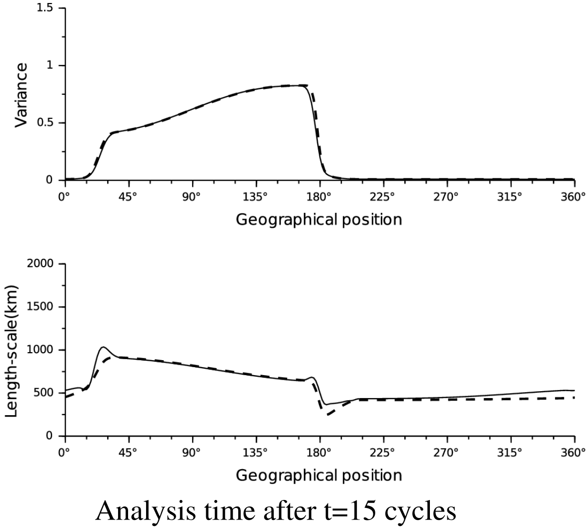


Figure: -2- Analysis covariance matrix: KF (continuous line), PKF (dashed line).

Simple analysis/forecast cycle



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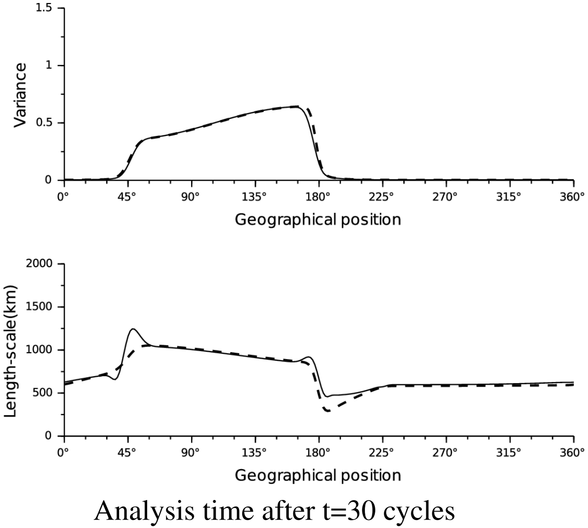

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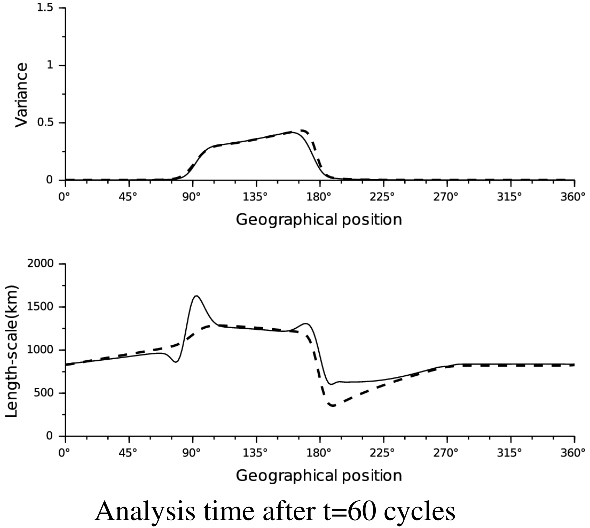

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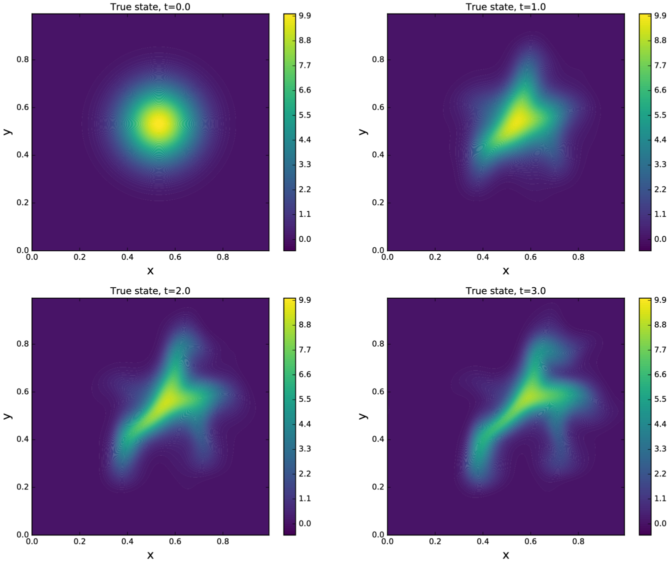
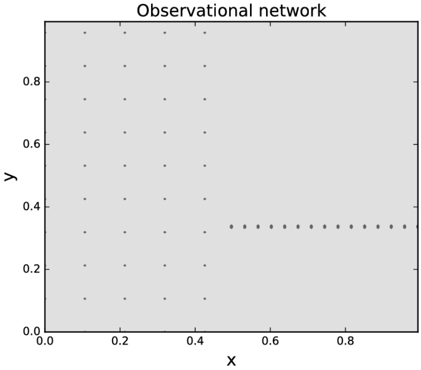
Analysis/forecast cycles, passive tracer: EnKF vs. PKF (dt=0.25)



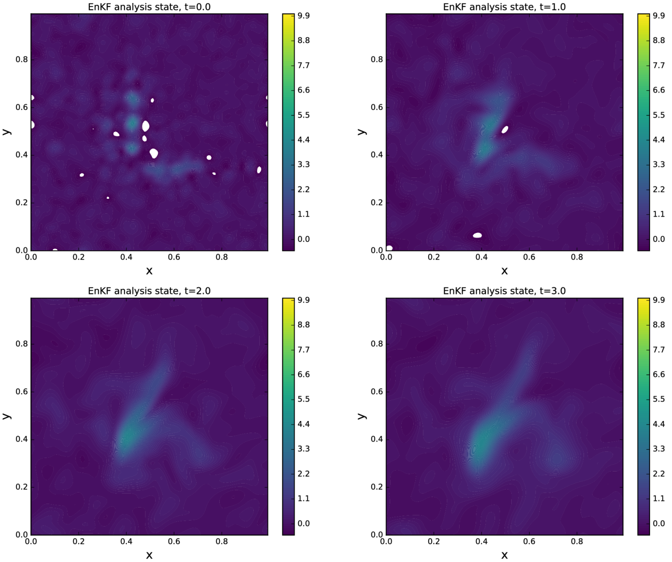
Advection-diffusion of a passive tracer

Analysis/forecast cycles, passive tracer: EnKF vs. PKF (dt=0.25)



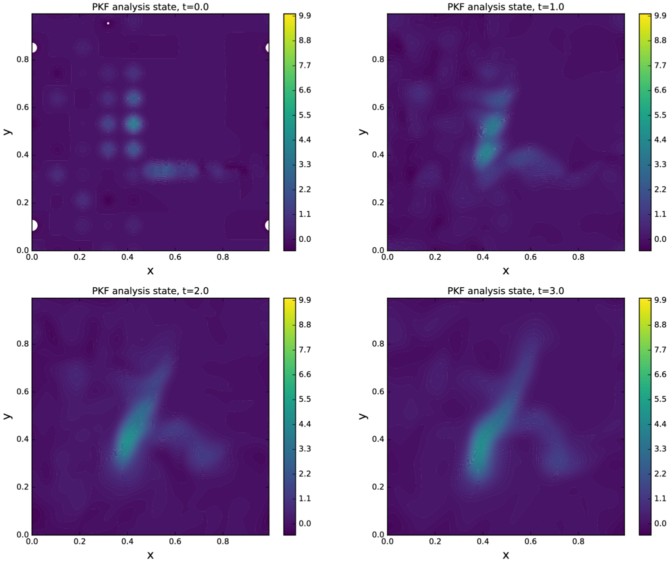
Regular network (left side) and simulated flight tracks (right side).

Analysis/forecast cycles, passive tracer: EnKF vs. PKF (dt=0.25)



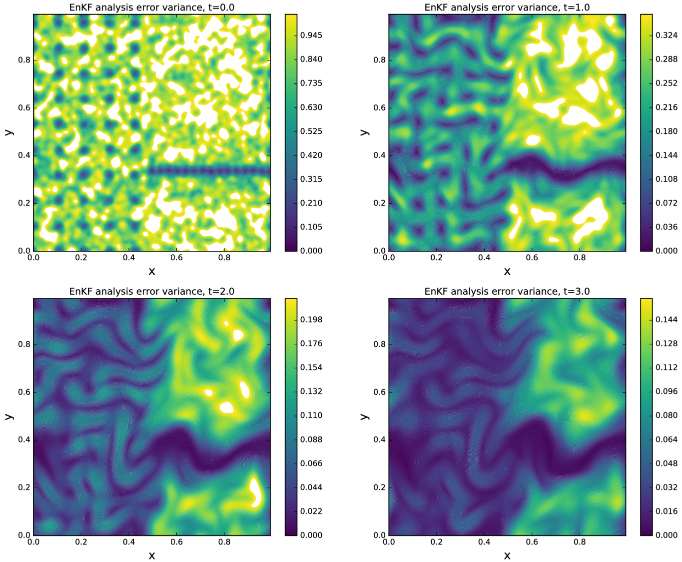
EnKF Analysis state (EnKF == $Ne = 100$ + EnVar + Localization)

Analysis/forecast cycles, passive tracer: EnKF vs. PKF (dt=0.25)



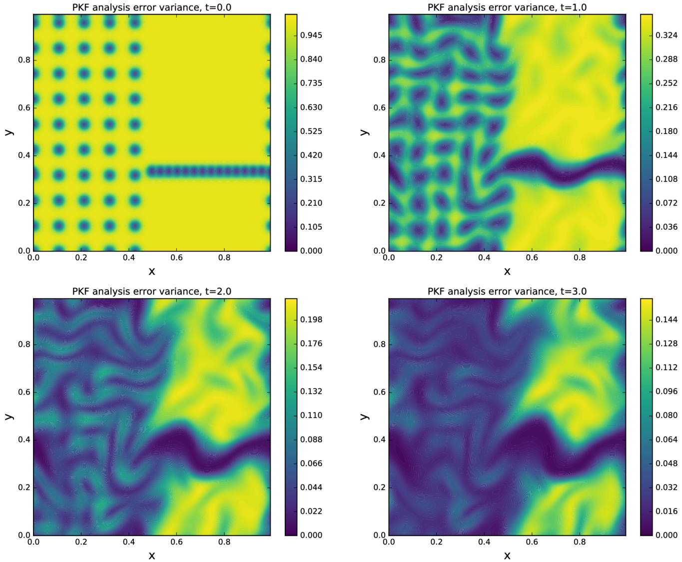
PKF Analysis state

Analysis/forecast cycles, passive tracer: EnKF vs. PKF (dt=0.25)



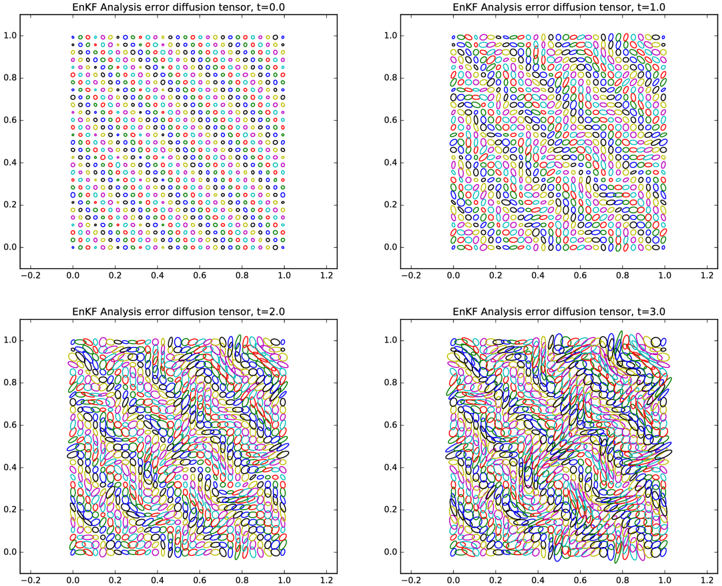
EnKF Analysis error variance fields after analysis/forecast cycles (EnKF == $Ne = 100$ + EnVar + Localization)

Analysis/forecast cycles, passive tracer: EnKF vs. PKF (dt=0.25)



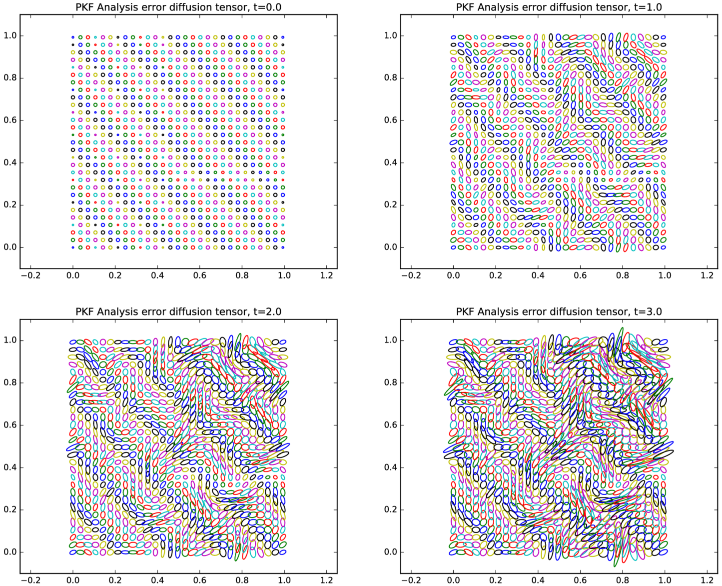
PKF Analysis error variance fields after analysis/forecast cycles

Analysis/forecast cycles, passive tracer: EnKF vs. PKF (dt=0.25)



EnKF Analysis error local diffusion fields after analysis/forecast cycles (EnKF == $Ne = 100$ + EnVar + Localization)

Analysis/forecast cycles, passive tracer: EnKF vs. PKF (dt=0.25)



PKF Analysis error local diffusion fields after analysis/forecast cycles

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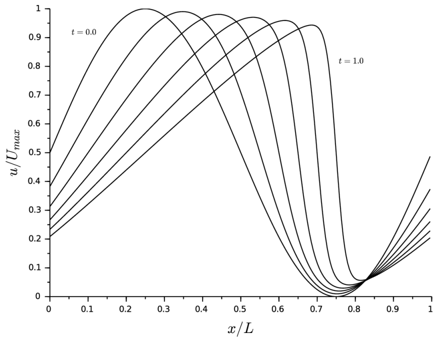
Diffusive non-linear Burgers dynamics

Burgers equation writes

$$\partial_t u + u\partial_x u = \kappa\partial_x^2 u. \quad (16)$$

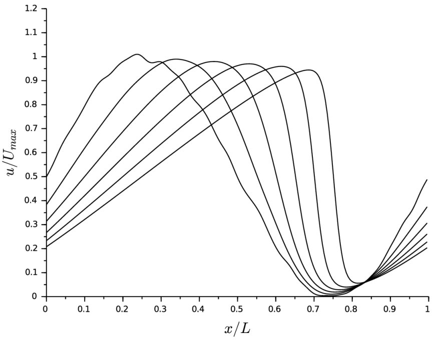


Solution starting from a cosine function.

Tangent-linear dynamics

With $u = \bar{u} + \varepsilon$, where $\bar{\cdot} \equiv \mathbb{E}[\cdot]$, the tangent-linear dynamics for small perturbations writes

$$\begin{cases} \partial_t \bar{u} + \bar{u}\partial_x \bar{u} = \kappa \partial_x^2 \bar{u} - \overline{\varepsilon \partial_x \varepsilon}, \\ \partial_t \varepsilon + \bar{u}\partial_x \varepsilon = -\varepsilon \partial_x \bar{u} + \kappa \partial_x^2 \varepsilon. \end{cases} \tag{17}$$

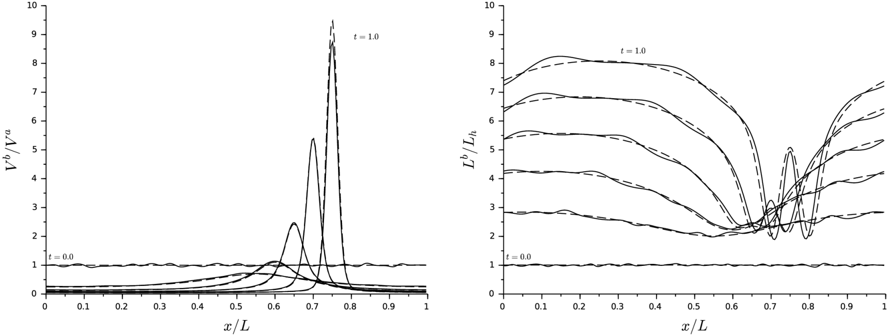


PKF forecast dynamics for diffusive Burgers

Then, dynamics of parameters writes

$$\begin{cases} (a) & \partial_t \bar{u} + \bar{u}\partial_x \bar{u} = \kappa \partial_x^2 \bar{u} - \frac{1}{2}\partial_x V, \\ (b) & \partial_t V + \bar{u}\partial_x V = -2(\partial_x \bar{u})V + \kappa \partial_x^2 V_x - \frac{\kappa}{2}\frac{1}{V_x}(\partial_x V_x)^2 - \frac{\kappa}{\nu_x} V_x, \\ (c) & \partial_t \nu_x + \bar{u}\partial_x \nu_x = 2(\partial_x \bar{u})\nu_x + 2\kappa - 2\frac{\kappa}{V_x}\partial_x^2 V_x \nu_x + 2\frac{\kappa}{V_x^2}(\partial_x V_x)^2 \nu_x + \\ & \qquad 2\kappa \frac{1}{V_x}\partial_x V_x \partial_x \nu_x + \kappa \partial_x^2 \nu_x - 2\kappa \frac{1}{\nu_x}(\partial_x \nu_x)^2. \end{cases} \tag{18}$$

which extends [Cohn, 1993] to the diffusive case.



(Error variance field) (Error length-scale field)

Ensemble (continuous line) vs. Parametric (dashed line)

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Conclusion

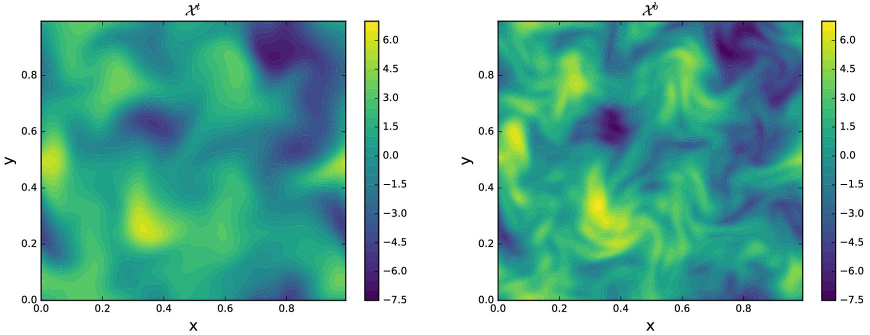
The Parametric Kalman Filter relies on covariance model to reproduce the uncertainty dynamics all along analysis/forecast cycles:

- no sampling noise,
- no localization,
- low numerical cost,
- relies on a given covariance model,
- needs developing dynamical equations for parameters,
- provide a new tool for understanding covariance dynamics for partial differential dynamics along analysis/forecast cycles

Further directions:

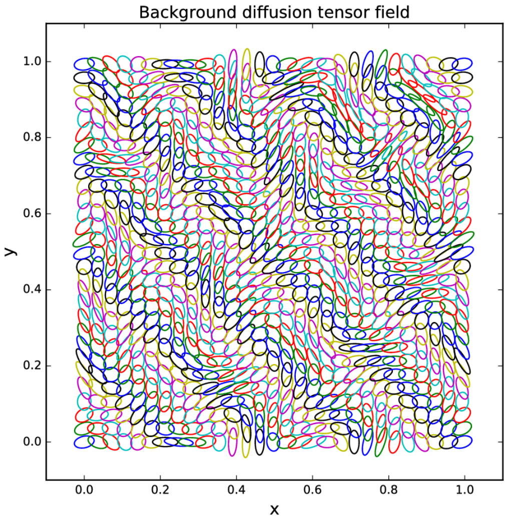
- applications for chemical transport model,
- extension for ocean/atmosphere dynamics.

Additionnal results – analysis in 2D: KF vs. PKF

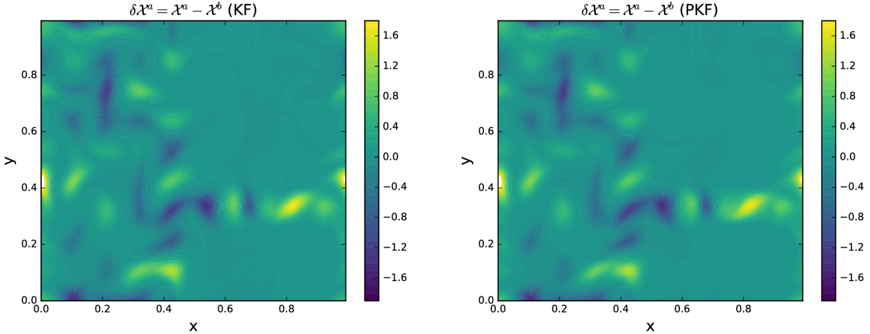


True state / Background state

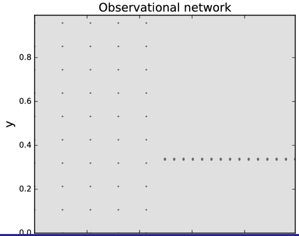
Additionnal results – analysis in 2D: KF vs. PKF



Forecast error diffusion field



Analysis increment KF (left) versus PKF (right)



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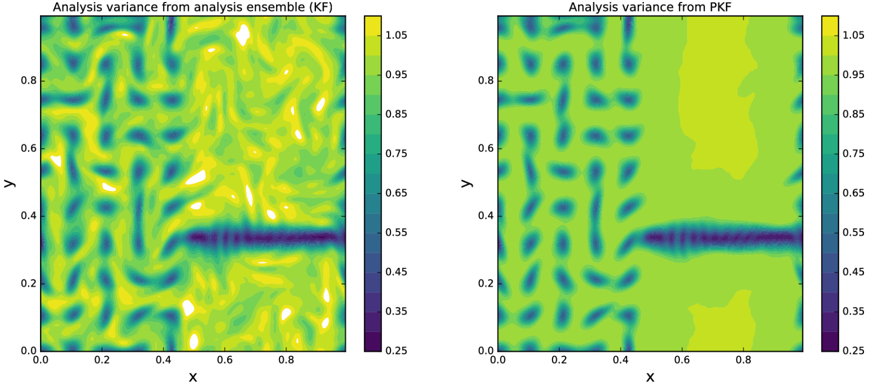

Analysis error variance fields: KF (left, estimated from 3DVar ensemble with pertubed observations, $Ne = 1600$) vs. PKF (right)

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