Multi-level ensemble based data assimilation

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Introduction

Coarse scale and multilevel ensemble based data assimilation

In reservoir simulation models

$$\text{state} = g(m) = (Saturation, Pressure)$$

obtained at a high computational cost

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by Monte-Carlo estimation

Quality of estimation relies on the ensemble size

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Quality of estimation relies on the ensemble size

Finite computational resources $\rightarrow$ ensemble size not optimal

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- Proxy model with adjustable accuracy:
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Earlier results: *Coarse scale ensemble based data assimilation*

A better balance between numerical and statistical accuracy results in improved DA results

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However:
High proxy error $\rightarrow$ Poor estimation of $p(m|d)$

Introduction
Multilevel methods

Problem with coarse scale DA: select accurate proxy model

The multilevel approach removes this problem

Alternative approach $\rightarrow$

1. MLEnKF – unbiased
2. Bayesian model average – biased

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Firstly: Investigate Multilevel Monte Carlo (MLMC)

Multilevel Monte Carlo

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$$\mathbb{E}[P_L] = \mathbb{E}[P_0] + \sum_{l=1}^{L} \mathbb{E}[P_l - P_{l-1}]$$

which can be estimated as

$$(N_e)_0^{-1} \sum_{n=1}^{(N_e)_0} P_0^n + \sum_{l=1}^{L} (N_e)_l^{-1} \sum_{n=1}^{(N_e)_l} (P_l^n - P_{l-1}^n)$$

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With unlimited computational resources $\rightarrow$ the ML method is more efficient than the standard MC method

Multilevel Monte Carlo

Application with restricted computational resources

MSE of MLMC for Euler discretisation of simple SDE [1]

$$c_2 h_L^2 + \sum_{l=0}^{L} c_1 (N_e)_l^{-1} h_l$$

where h_l is grid size at level l, c_1 and c_2 weights bias and variance

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MLMC fails when
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In general c_1 and c_2 are unknown

---

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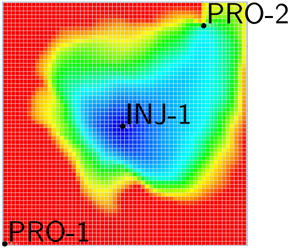
Two phase flow test

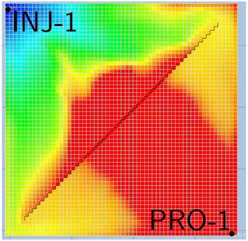
Investigate two test cases

- 60×60 grid-cells
- 80 assimilation time steps
- Proxy via uniform upscaling
- Model 1: No fault
- Model 2: Dominant impermeable fault

- Model 1 & 2: computational resources = 10 full runs



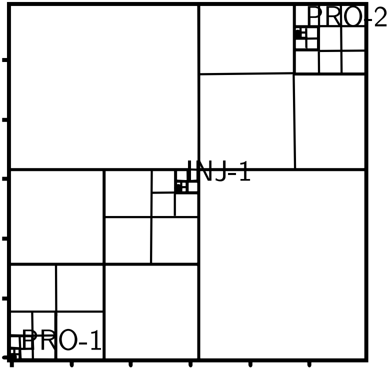
Model 1



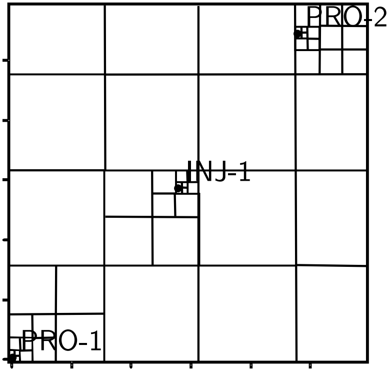
Model 2

Uniform upscaling

Grid model 1



Level 0



Level 1

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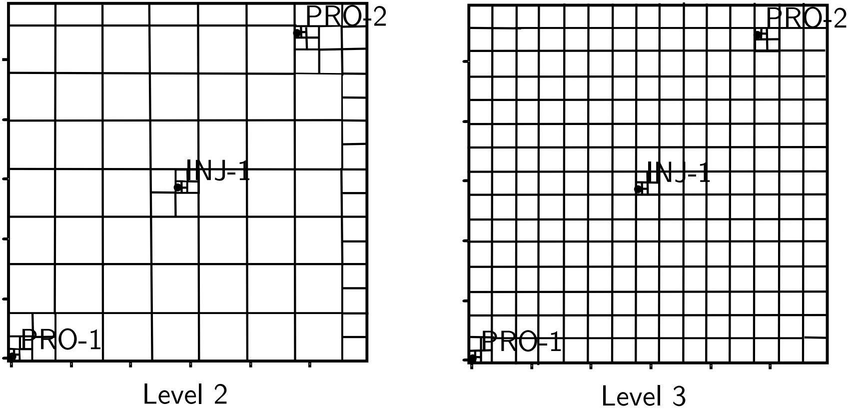

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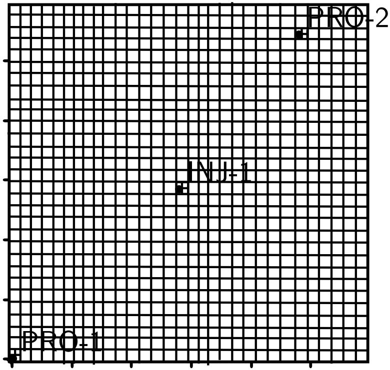


Level 4

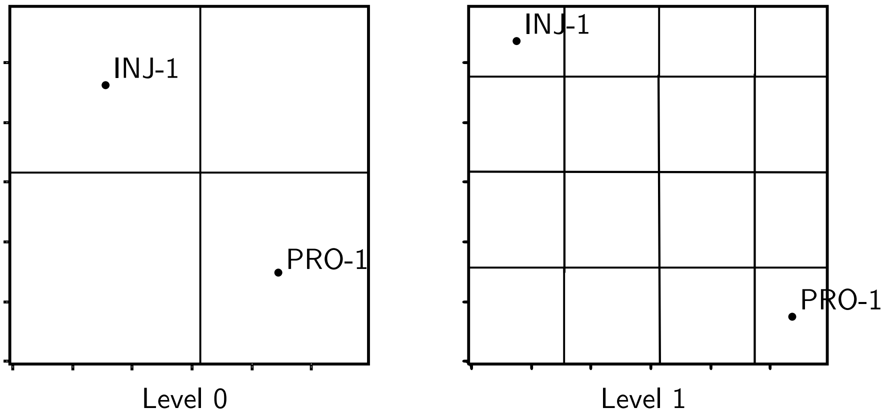
Grid-size for the various levels

Level 0	47
Level 1	53
Level 2	98
Level 3	243
Level 4	909

Original grid: 3600 grid-cells

Uniform upscaling

Grid model 2



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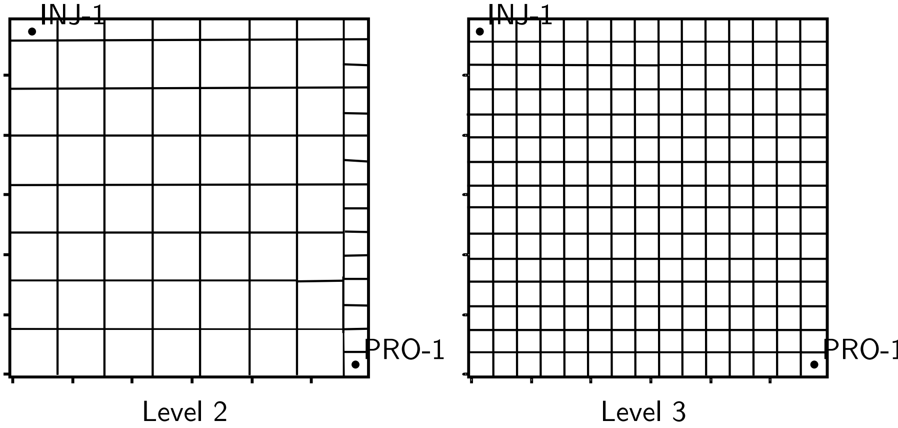

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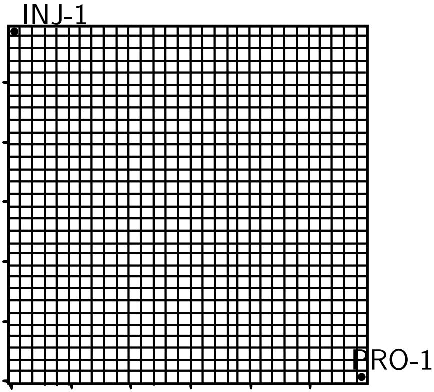


Level 4

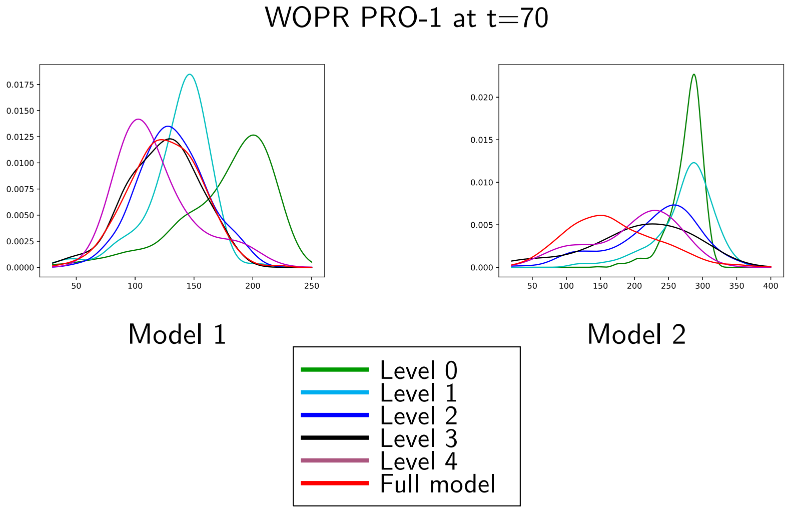
Grid-size for the various levels

Level 0	4
Level 1	16
Level 2	71
Level 3	225
Level 4	900

Original grid: 3600 grid-cells

Uniform upscaling

Kernel density estimate of simulator output



Bias and variance of C_{mg}

Estimation by bootstrapping

For data assimilation applications estimate of C_{mg} is important

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Evaluate the bias and variance for $(C_{mg})_l$ for $l = 0, 1, 2, 3, 4$ by bootstrapping

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Define $(N_e)_l = (N_e^{\text{full}})_l \times \frac{N_g^{\text{full}}}{(N_g)_l}$
where $\sum_{l=0}^{4} (N_e^{\text{full}})_l = 10$

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Ensemble size for the various levels

	Level 0	Level 1	Level 2	Level 3	Level 4
Model 1	454	209	123	43	9
Model 2	900	500	102	32	12

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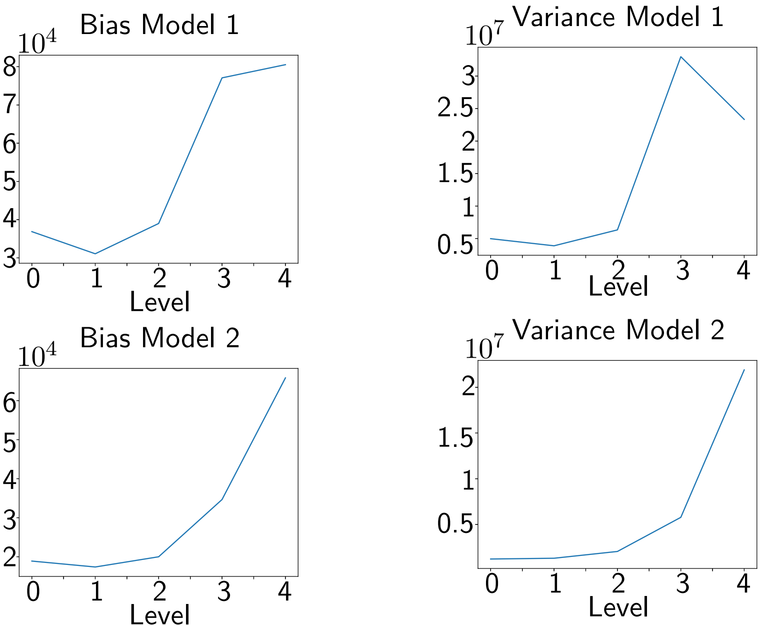
Calculate bias and variance of $(C_{mg})_l$ by 2000 replications

Element wise bias and variance of C_{mg}

Frobenius norm



Bias and variance

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Keeping the total computational resources fixed on each level we observe that for both models

– Variance is the dominant factor
– Bias *increases* with accuracy due to MC error

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– Realistic cases have lower value of $^{\text{computational resources}}/_{\text{grid-cells}}$

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Analysis of Model 1 & 2 with limited computational resources
– Resources best spent to reduce variance

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– Resources best spent to reduce variance

$\rightarrow$ Evaluate two different multilevel algorithms

Multilevel methods

Multilevel EnKF

MLMC can be extended to the EnKF framework

$$C_{ML} = C_0 + \sum_{l=1}^{L} (C_l - C_{l-1})$$

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the MLEnKF analysis is then

$$m^a = m^f + C_{ML} H^T (H C_{ML} H^T + C_d)^{-1} (d - g(m))$$

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– Converges to the KF solution

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Bayesian model average

Let the forecast density be defined by Bayesian model averaging

$$p(Y|d) \propto p(d|Y)p(Y) = p(d|Y)\sum_{l=0}^{L} p(Y|M_l)p(M_l)$$

Each model M_l represents an accuracy level of the proxy

Assume that all densities are Gaussian

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Bayesian model averaging utilize all proxy models

– Bias-variance tradeoff adjusted through the weights, $p(M_l)$

Multilevel methods

Bayesian model average

New total empirical forecast covariance given by law of total covariance

- $C_{tot} = \sum_{l=0}^{L} p(M_l) C_l + \sum_{l=0}^{L} p(M_l)(\mu_l - \overline{\mu})(\mu_l - \overline{\mu})^T$

How to select $p(M_l)$?

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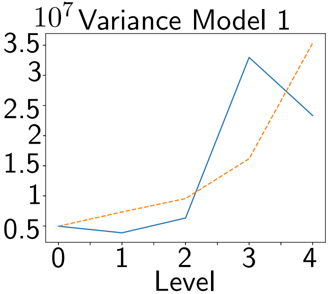

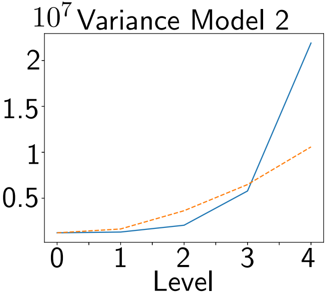

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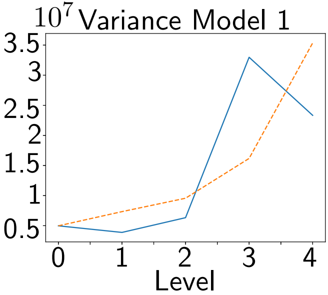

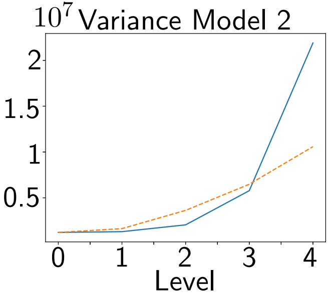

Dashed line: $\frac{c}{\sqrt{(N_e)_l}}$

Select the weights as $p(M_l) \propto \sqrt{(N_e)_l}$
– constraints $\sum_{l=0}^{L} p(M_l) = 1$

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Bayesian model average – analysis step

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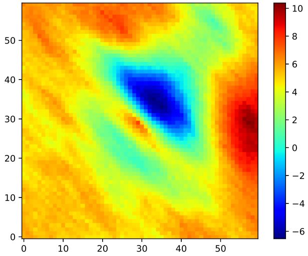
$\rightarrow$ if all models have same accuracy $\alpha_l = 1 \quad \forall l$

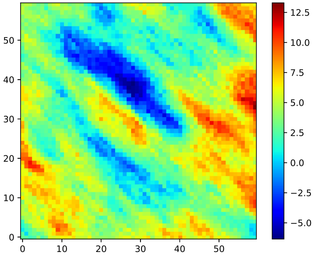
In the Gaussian case with linear dynamic models:
– Equally accurate models $\rightarrow$ Converge to KF

BMA

Mean example 1



ES with large ensemble



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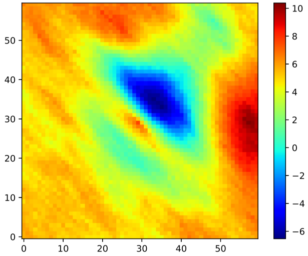

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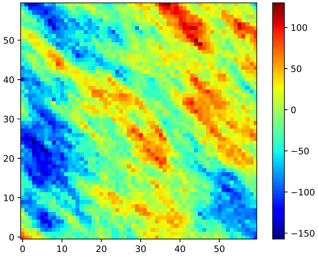
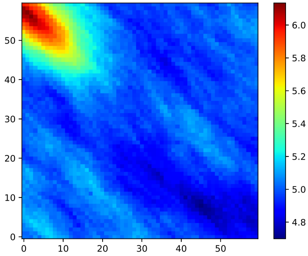


MLEnKF

BMA

Mean example 2



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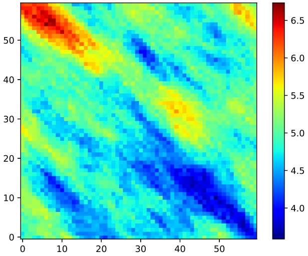

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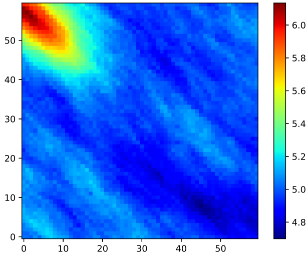

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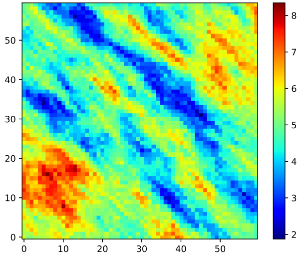

MLEnKF

Summary and conclusion

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An alternative multilevel method, based on BMA is introduced to handle such cases
– Method aims to reduce variance
– Method is biased

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MLMC methods are optimal for cases that are not dominated by variance, and with unlimited computational resources

Analysis of two simple two-phase flow problems indicates that MLMC methods are not optimal for these cases

An alternative multilevel method, based on BMA is introduced to handle such cases
– Method aims to reduce variance
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Numerical DA experiments shows that
– MLEnKF fails to estimate the mean
– The alternative method gives good estimates of the mean

Acknowledgements

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