

CONSTITUTIVE ASSUMPTIONS

The ensemble is an iid sample from the same *Gaussian* distribution as the truth, i.e. $\mathbf{x}, \mathbf{x}_n \sim \mathcal{N}(\mathbf{b}, \mathbf{B})$. However, the true moments, \mathbf{b} and \mathbf{B} are *unknown*. Then, by marginalization,

$$p(\mathbf{x}|\mathbf{E}) = \int p(\mathbf{x}|\mathbf{b}, \mathbf{B}, \mathbf{E}) p(\mathbf{b}, \mathbf{B}|\mathbf{E}) d\mathbf{b} d\mathbf{B}.$$

THE ENKF- N EFFECTIVE PRIOR

With $p(\mathbf{b}, \mathbf{B}) \propto |\mathbf{B}|^{-(m+1)/2}$, the effective prior is

$$p(\mathbf{w}|\mathbf{E}) = \left(\varepsilon_N + \|\mathbf{w}\|_{\mathbf{I}_N}^2 \right)^{-(N+g)/2} \propto \mathbf{t}_N(\mathbf{w}|g; 0, \varepsilon_N \mathbf{I}_N),$$

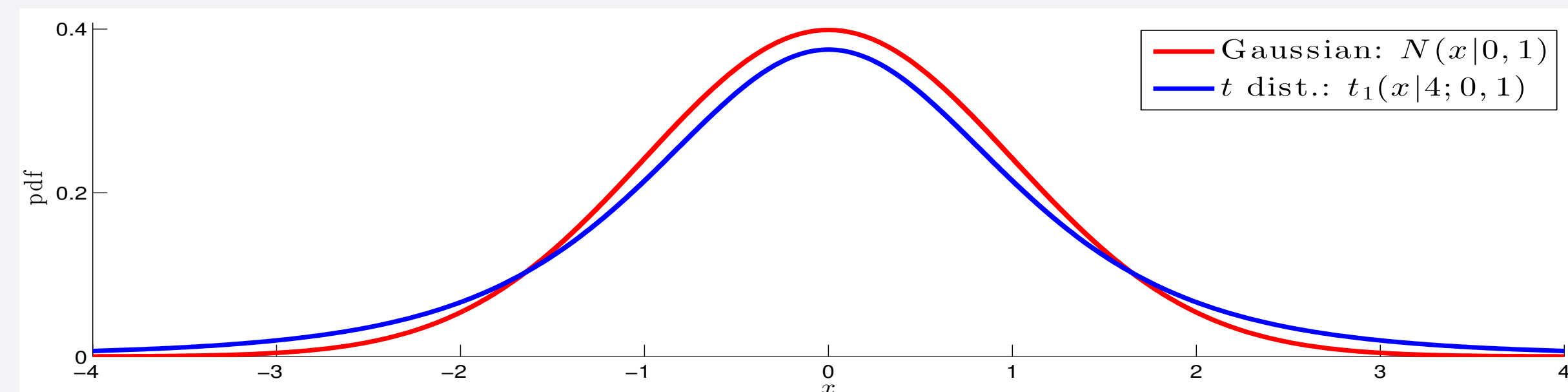
where $\varepsilon_N = 1 + \frac{1}{N}$, and g is the rank of the nullspace of \mathbf{A} . The posterior is $p(\mathbf{w}|\mathbf{E}, \mathbf{y}) \propto \exp -\frac{1}{2} (\|\mathbf{y} - h(\mathbf{x}(\mathbf{w}))\|_{\mathbf{R}}^2 + (N+g) \log(\varepsilon_N + \|\mathbf{w}\|_{\mathbf{I}_N}^2))$

THE t DISTRIBUTION

Note the thick tails.

$$\mathcal{N}(x|0, 1) \propto e^{-\frac{1}{2}x^2}$$

$$\mathbf{t}_1(x|1; 0, 1) \propto \frac{1}{1+x^2}$$



SCALE MIXTURE RE-DERIVATION

The dual EnKF- N may also be derived as a *scalar*, scale mixture. Let $\lambda^2 = \|\mathbf{x} - \bar{\mathbf{x}}\|_{\bar{\mathbf{B}}}^2 / \|\mathbf{x} - \bar{\mathbf{x}}\|_{\mathbf{B}}^2$. Then

$$p(\mathbf{w}|\mathbf{E}, \mathbf{y}) = \int p(\mathbf{w}, \lambda^2|\mathbf{E}, \mathbf{y}) d\lambda^2$$

$$\propto \int \mathcal{N}(\bar{\boldsymbol{\delta}}|\mathbf{Y}\mathbf{w}, \mathbf{R}) \mathcal{N}(\mathbf{w}|\lambda^2, \frac{\varepsilon_N}{N-1} \mathbf{I}_N) \chi^{-2}(\lambda^2|N-1) \lambda^{-g} d\lambda$$

$$\propto \int \exp -\frac{1}{2} \left(\|\mathbf{Y}\mathbf{w} - \bar{\boldsymbol{\delta}}\|_{\mathbf{R}}^2 + \frac{N-1}{\varepsilon_N \lambda^2} \|\mathbf{w}\|_{\mathbf{I}_N}^2 + \frac{N-1}{\lambda^2} + (N+g) \log \lambda^2 \right) d\lambda$$

$$\approx c \exp -\frac{1}{2} \left(\|\mathbf{w} - \mathbf{w}_*(\zeta_*)\|_{\mathbf{G}_*(\zeta_*)}^2 + \underbrace{\|\bar{\boldsymbol{\delta}}\|_{\mathbf{Y}\mathbf{Y}^\top/\zeta_* + \mathbf{R}}^2 + \varepsilon_N \zeta_* - (N+g) \log \zeta_*}_{D(\zeta_*)} \right),$$

where $\zeta = \frac{N-1}{\varepsilon_N \lambda^2}$.

THE STANDARD ENKF PRIOR

Denote $\mathbf{E} = [\mathbf{x}_1, \dots, \mathbf{x}_n, \dots, \mathbf{x}_N]$ and let \mathbf{A} be the corresponding matrix of anomalies. Parameterize \mathbf{x} by ensemble weights, \mathbf{w} :

$$\mathbf{x}(\mathbf{w}) = \bar{\mathbf{x}} + \mathbf{A}\mathbf{w}$$

$$\mathbf{y} - h(\mathbf{x}(\mathbf{w})) = \mathbf{y} - h(\bar{\mathbf{x}} + \mathbf{A}\mathbf{w}) \approx \bar{\boldsymbol{\delta}} - \mathbf{Y}\mathbf{w}$$

Then, $p(\mathbf{x}|\mathbf{b}=\bar{\mathbf{x}}, \mathbf{B}=\bar{\mathbf{B}}) \propto \exp(-\frac{1}{2}\|\mathbf{w}\|_{\mathbf{I}_N}^2)$.

THE ENKF'S INTRINSIC BIAS

Define $\bar{\mathbf{B}} = \frac{1}{N-1} \sum_{n=1}^N (\mathbf{x}_n - \bar{\mathbf{x}})(\mathbf{x}_n - \bar{\mathbf{x}})^\top$. Then

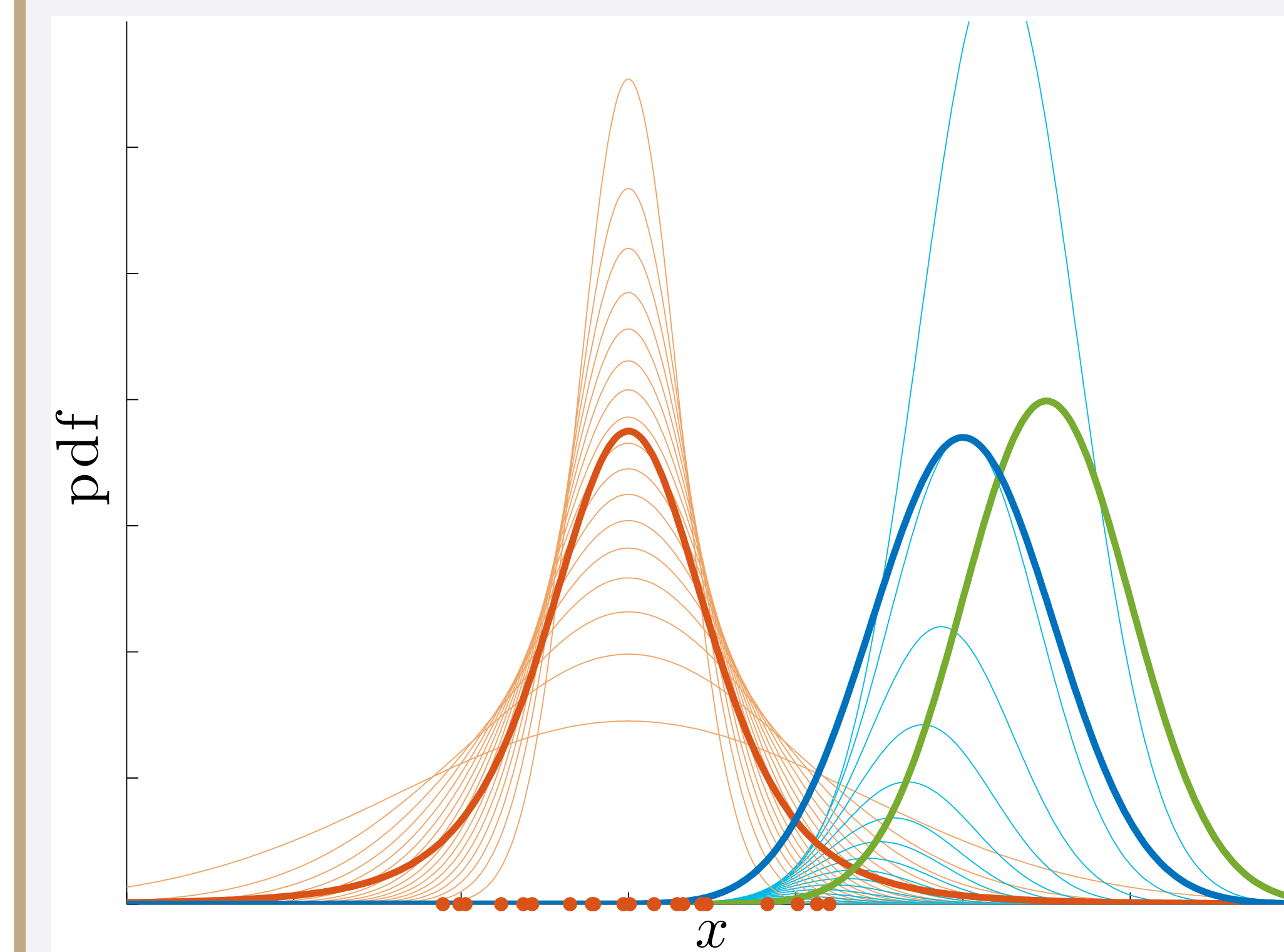
$$\mathbb{E}(\bar{\mathbf{B}}) = \mathbf{B},$$

(with expectation over all \mathbf{x}_n). But, with $\bar{\mathbf{P}}^a = [\mathbf{I} - \bar{\mathbf{K}}\mathbf{H}]\bar{\mathbf{B}}$, and the same expectation,

$$\mathbb{E}(\text{tr}(\bar{\mathbf{P}}^a)) < \text{tr}(\mathbf{P}^a),$$

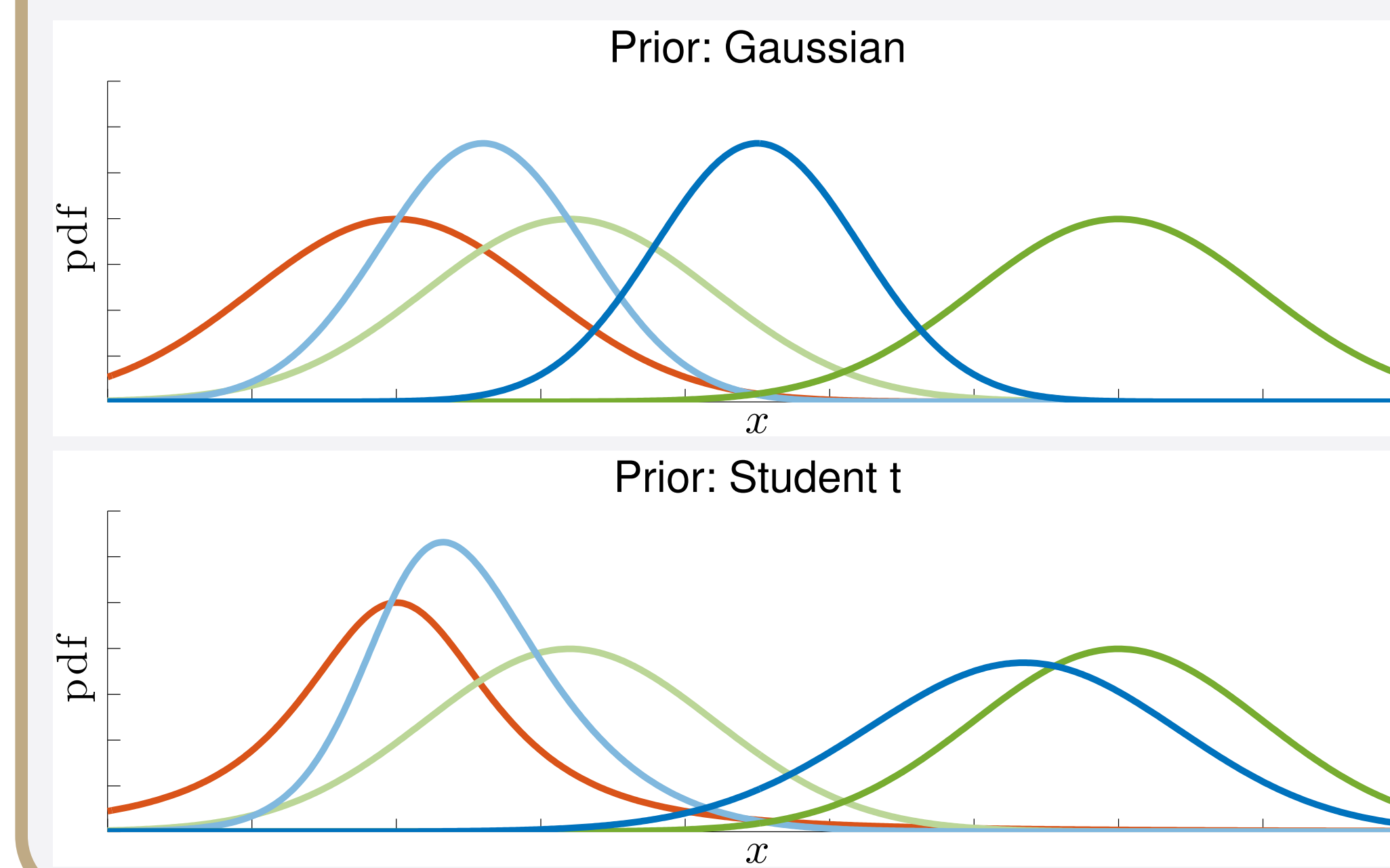
and idem. for $\bar{\mathbf{K}}$.

SCALE (AND LOC.) MIXTURES

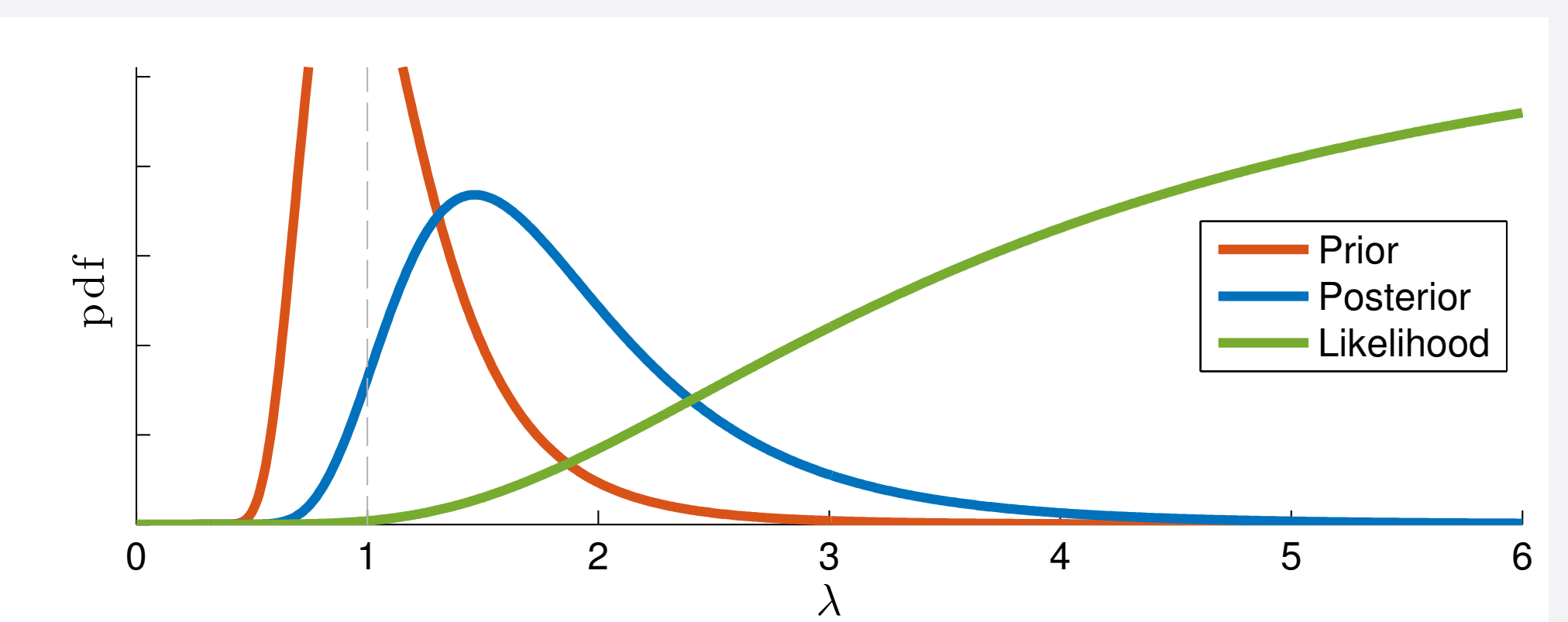


The effective (thick lines) prior and posteriors are the averages of the "candidate" (thin lines) prior and posteriors.

POSTERIOR'S BEHAVIOUR



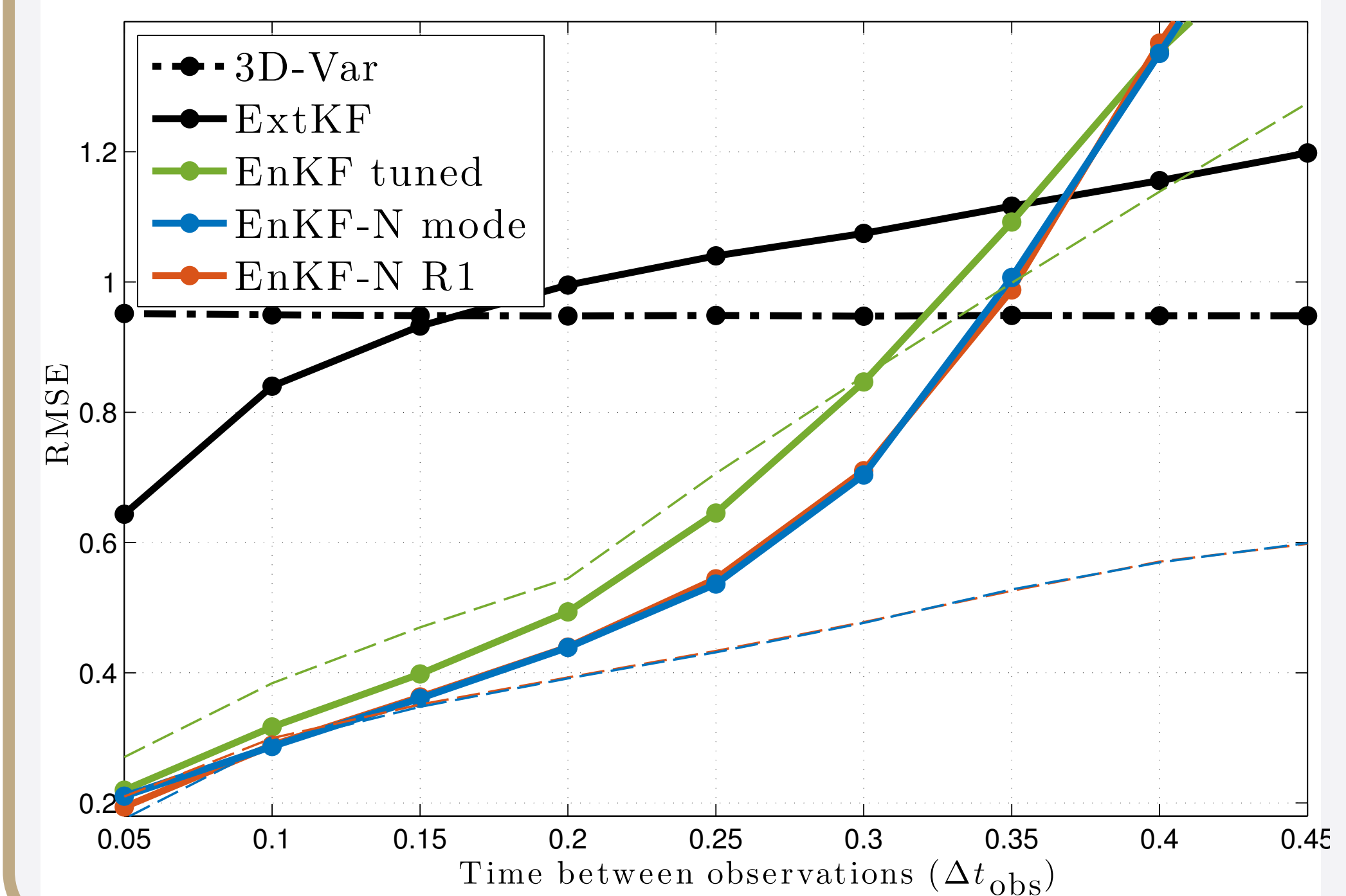
DISTRIBUTIONS OF λ^2



Prior: $p(\lambda^2|\mathbf{E}) = \chi^{-2}(\lambda^2|N-1)$
 Lklhd: $p(\mathbf{y}, \mathbf{w}_*|\mathbf{E}, \lambda^2) = \exp(-\frac{1}{2}\|\bar{\boldsymbol{\delta}}\|_{\mathbf{Y}\mathbf{Y}^\top/\zeta + \mathbf{R}}^2)$
 Post: $p(\mathbf{w}_*, \zeta|\mathbf{E}, \mathbf{y}) = \exp(-\frac{1}{2}D(\zeta))$

BENCHMARKS

Lorenz-96, $N = 20$, no model noise, $\mathbf{R} = \mathbf{I}_{40}$, no localization.



REFERENCES

- [1] Patrick Nima Raanes. *Improvements to Ensemble Methods for Data Assimilation in the Geosciences*. PhD thesis, University of Oxford, Mathematical Institute, 2015. Chapter 6.
- [2] Marc Bocquet, Patrick N. Raanes, and Alexis Hanhart. Expanding the validity of the ensemble Kalman filter without the intrinsic need for inflation. *Nonlinear Processes in Geophysics*, 22(6):645–662, 2015.

ACKNOWLEDGEMENTS



SUMMARY

- Less dogmatic assumptions \implies EnKF- N
 - Posterior variance depends on innovation
 - Better than "unbiased"
 - Sequential feedback
 - Careful about parameterization and implicit assumptions
- Primal perspective: Student t prior
 - Future: include localization
- Dual perspective: scale mixture
 - Adaptive inflation
 - Future: estimate model error inflation