# Bayesian data assimilation for cardiovascular flow using bi-fidelity Ensemble Kalman Inversion

**Pan Du,** Han Gao, Jian-Xun Wang*
*Department of Aerospace and Mechanical Engineering*
*University of Notre Dame*
http://sites.nd.edu/jianxun-wang/

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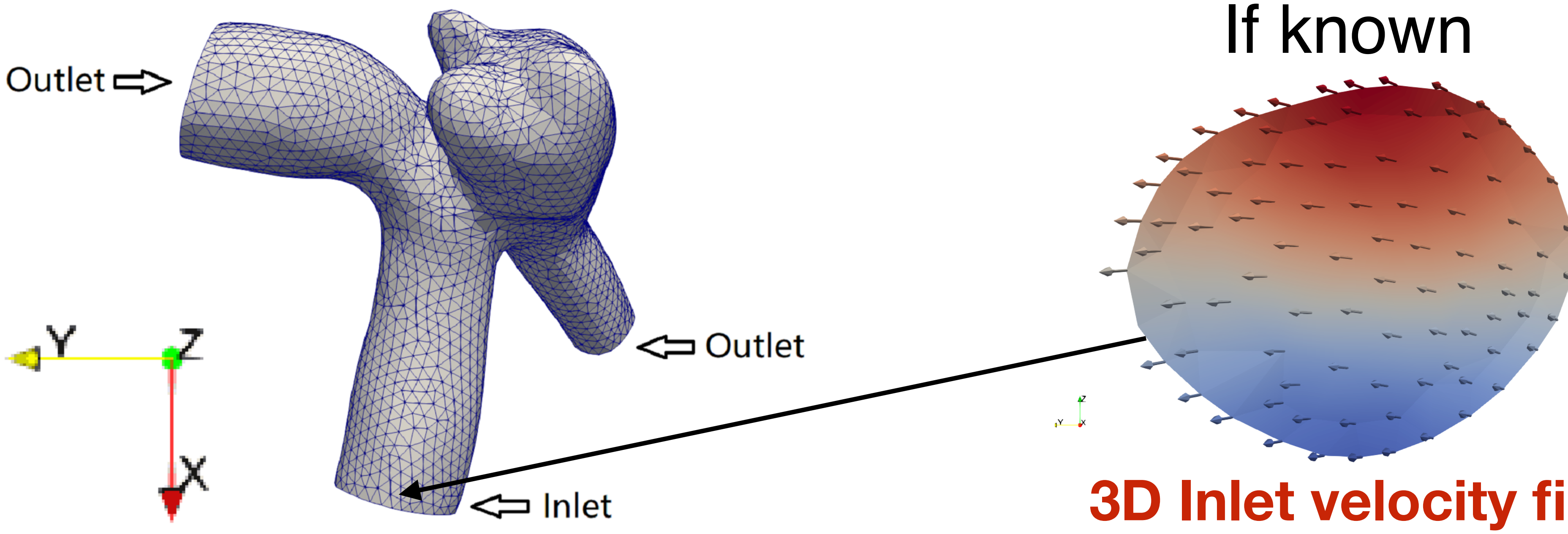
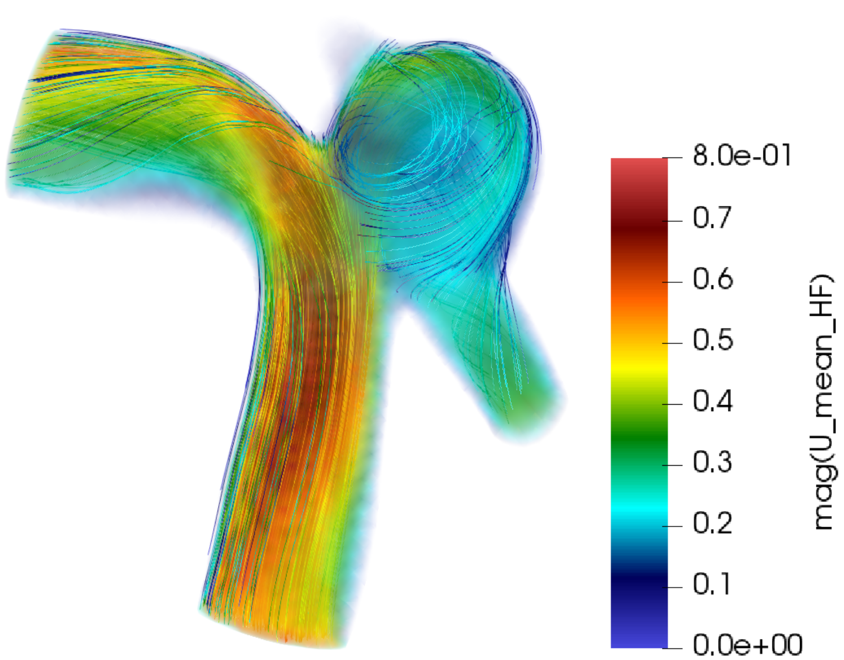
# Bi-fidelity Ensemble Kalman Inversion

**Example**: Inlet inversion in aneurysm bifurcation

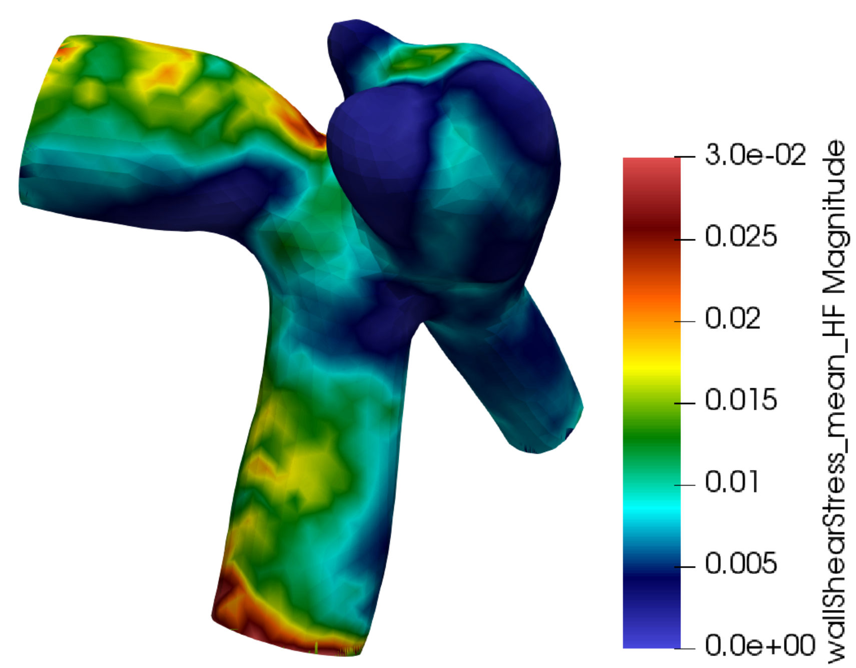
If known




3D Inlet velocity field



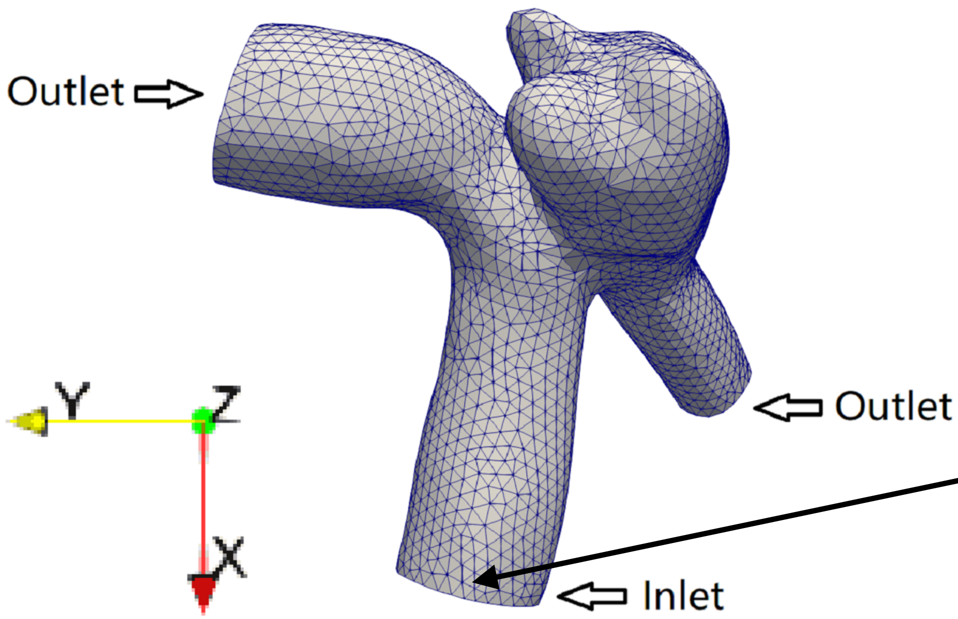

Flow field




Wall shear stress

# Bi-fidelity Ensemble Kalman Inversion

**Example**: Inlet inversion in aneurysm bifurcation




Not known

**3D Inlet velocity field**

**Infer**

**Sparse, noisy flow measurements**

**Example**: Inlet inversion in aneurysm bifurcation

Not known

$\boldsymbol{\theta} \xrightarrow[\text{CFD model}]{\mathbf{F}(\boldsymbol{\theta})} \boldsymbol{y}$

3D Inlet velocity field

Sparse, noisy flow measurements (Data)

$$\hat{x} = F(\theta, x_0) = \tilde{x} + \sigma_m$$

$$y = H(\tilde{x}) + \sigma_d$$

**Assimilate sparse data into model**

$$P(\hat{x}, \theta \,|\, y) \sim P(\theta)P(y \,|\, \theta)$$

**Bayes rule**

# Approximate Bayesian: ensemble Kalman update

**Assumptions**

Linear forward

$$\hat{x}_t = F(x_{t-1}) + \sigma_m$$

Gaussian Prior

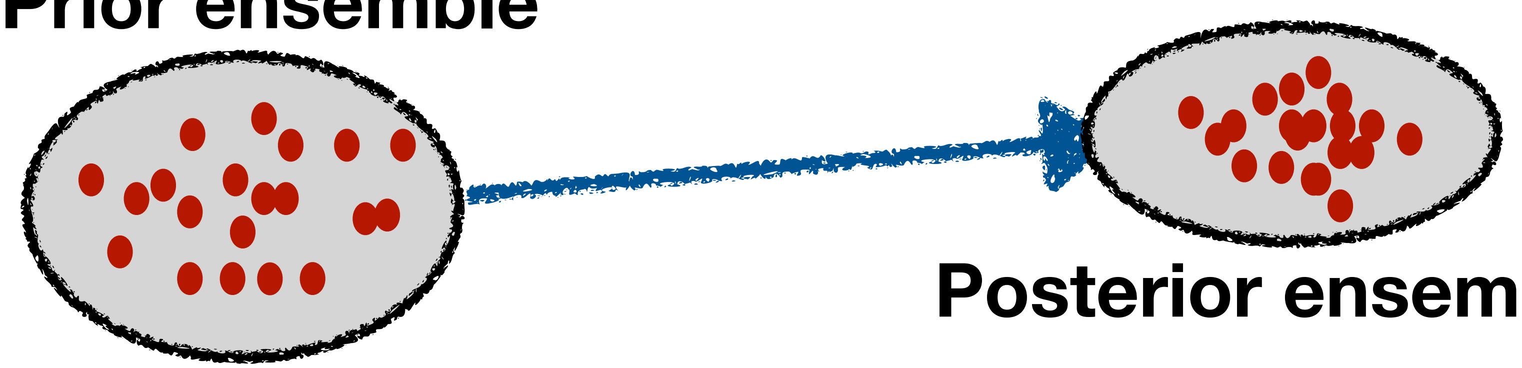
$$y_t = H\tilde{x}_t + \sigma_d$$

Gaussian Likelihood

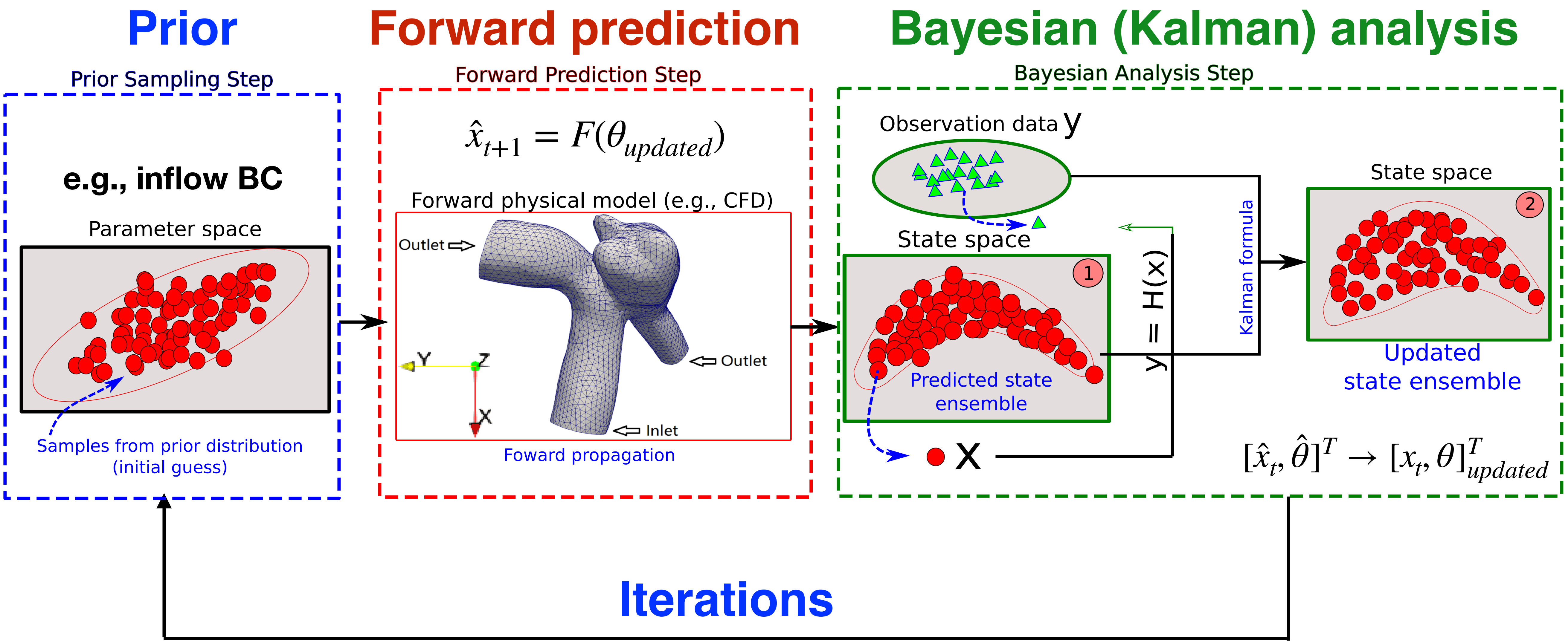
**Posterior**

$$x_t^{(j)} = \hat{x}_t^{(j)} + C_{\hat{x}_t\hat{x}_t} H^T \left(H C_{\hat{x}_t\hat{x}_t} H^T + \Gamma\right)^{-1} \left(y_t^{(j)} - H\hat{x}_t^{(j)}\right)$$

Kalman update formula for state estimation



# Iterative Ensemble Kalman Inversion

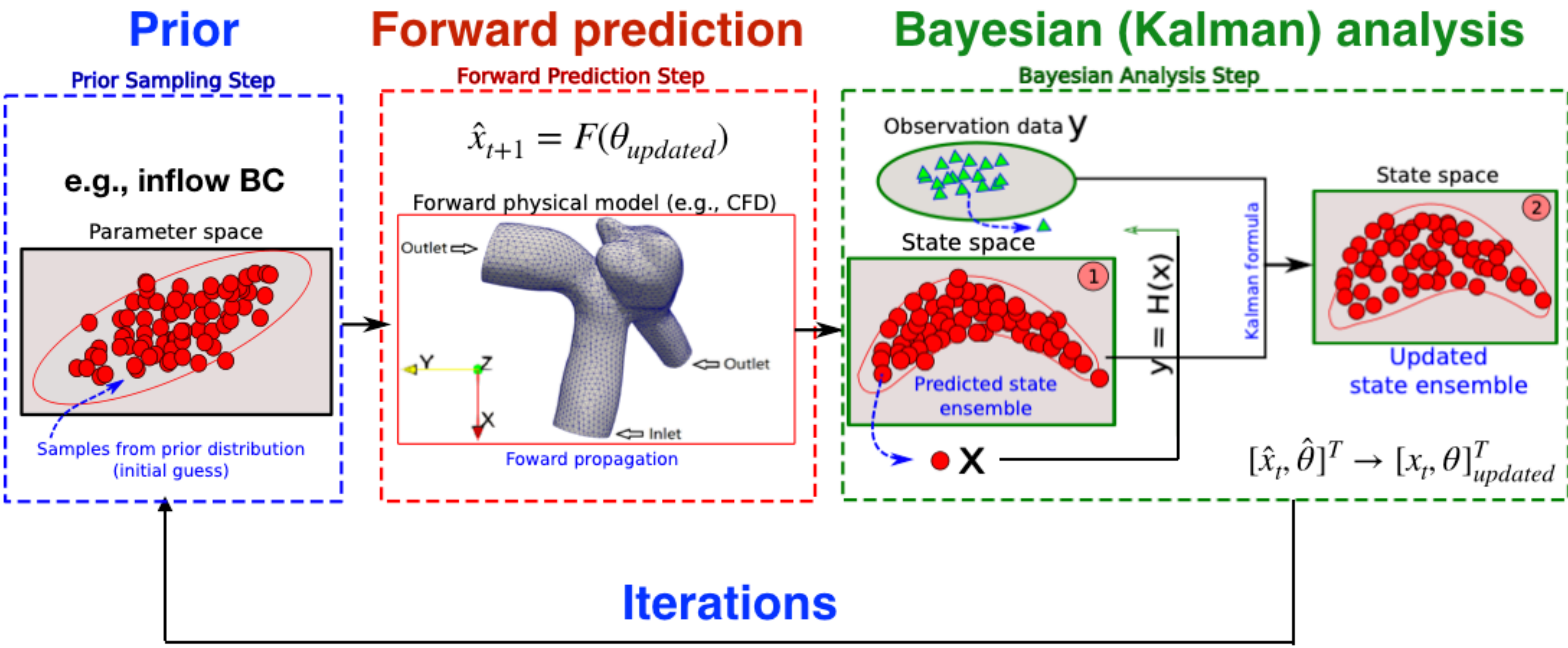


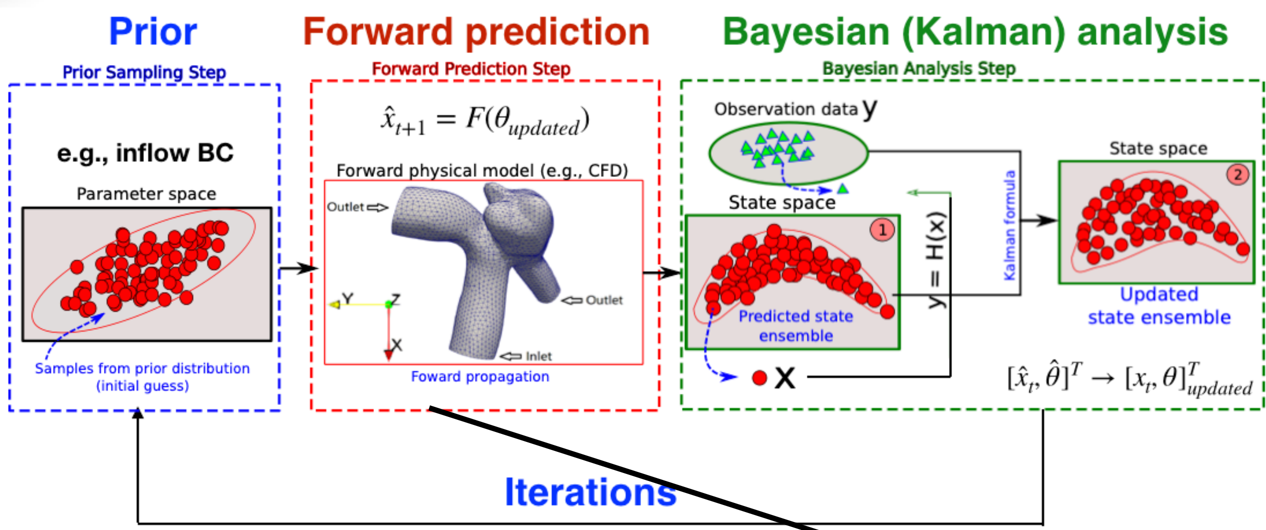

Proposed by M. Iglesias, A. Stuart, 2013, 2017, 2018

**This is also can be roughly formulated as a constrained optimization problem**

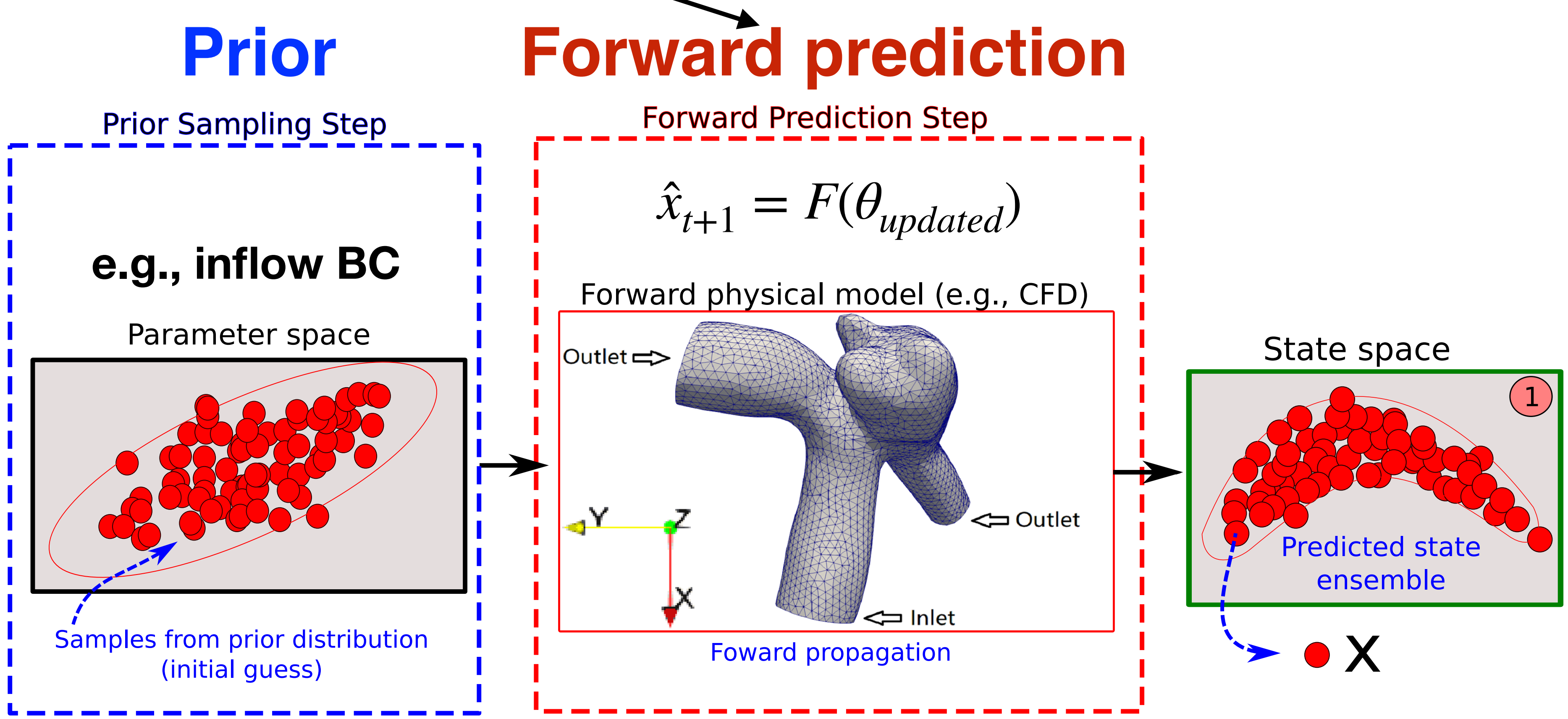
$$\min_{\theta,x} \|\bar{y} - Hx\|^2, \quad s.t. \quad x = F(\theta)$$

# Computational efficiency of Iterative EKI



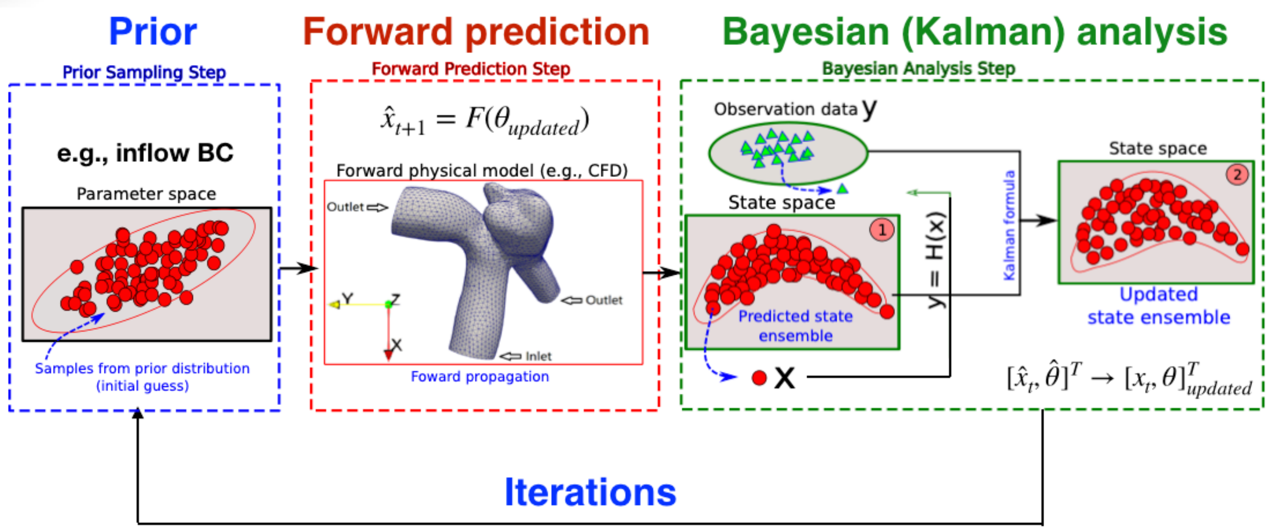



# How to reduce the computational cost?

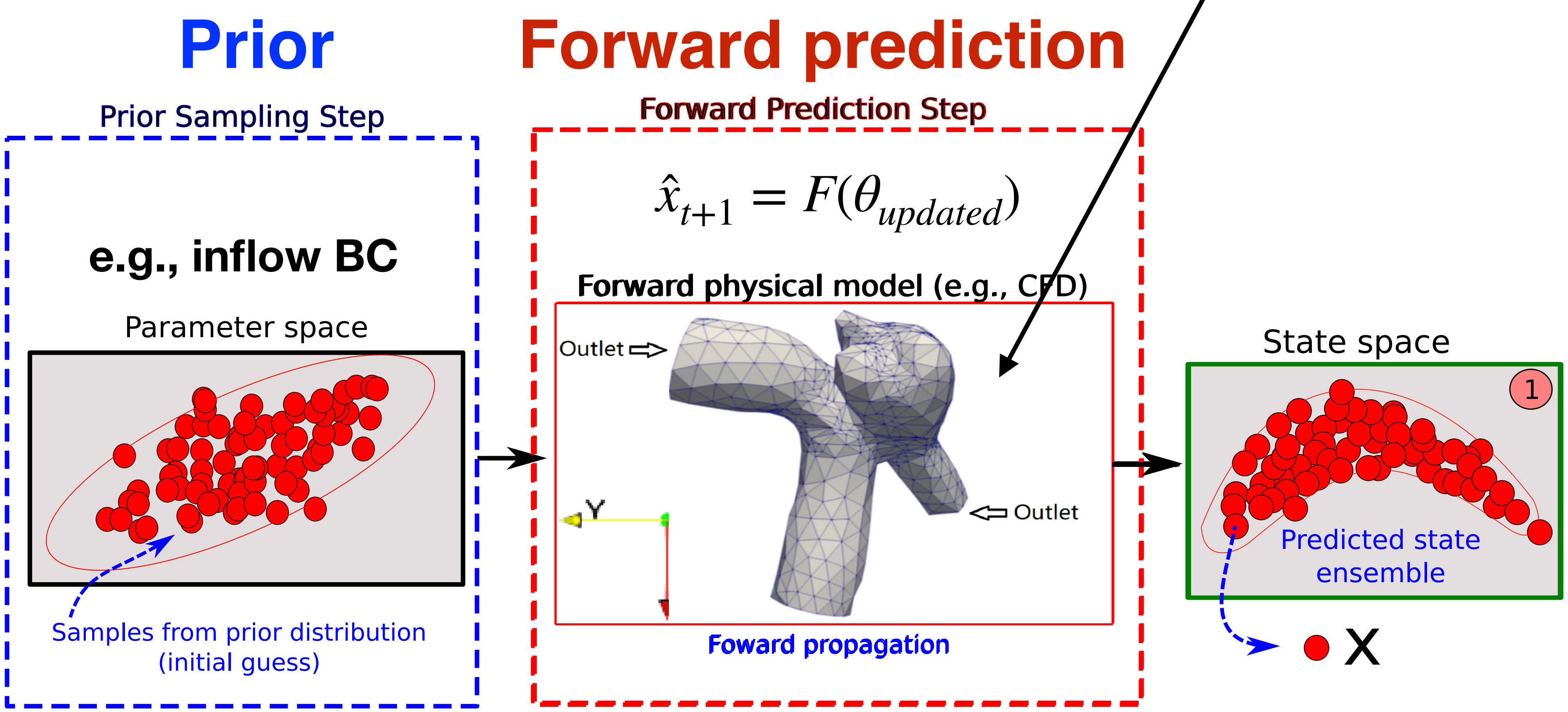



**Many evaluations of forward model, so if forward evaluation is expensive # of iteration * # of samples —> Computationally prohibitive!**

# Computational efficiency of Iterative EKI




**Replace the hi-fidelity forward model with the low-fidelity model**

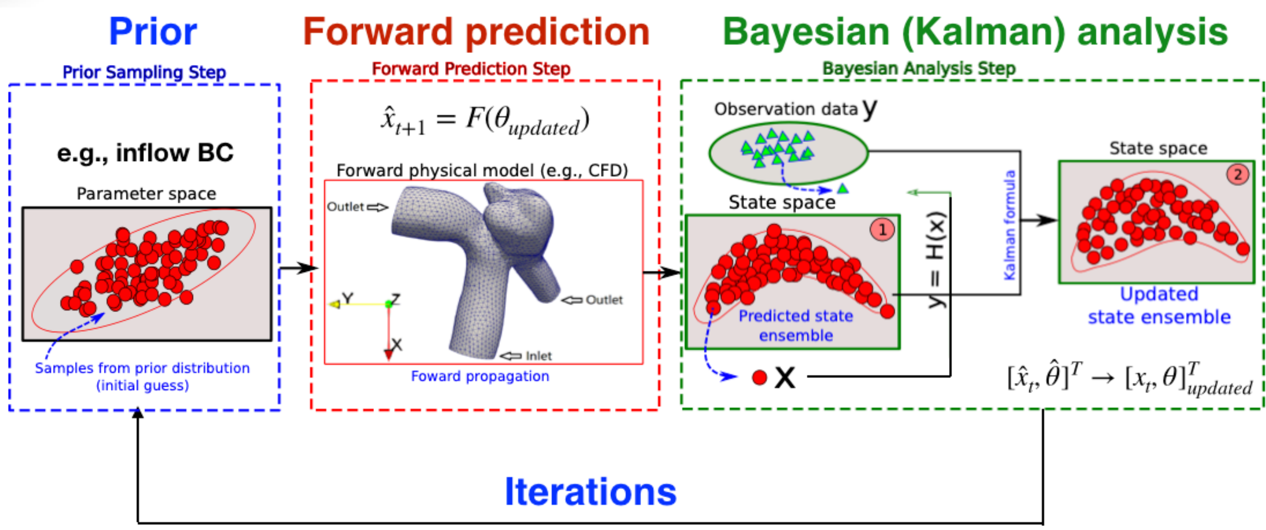



low-fidelity model (e.g., coarse mesh, unconverged solution)

hi-fidelity model (e.g., fine mesh, converged solution)

**However, low-fidelity model predictions has low accuracy, and thus inferred results could be very wrong!**

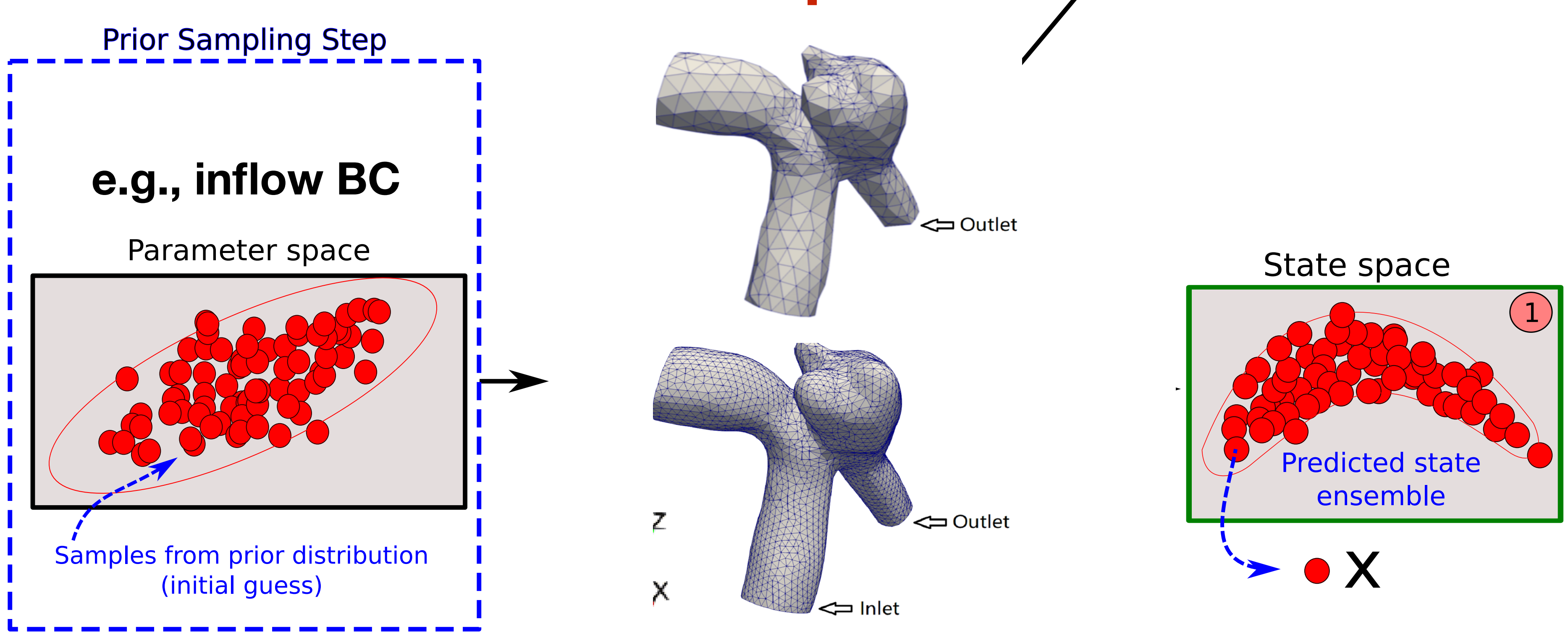
# Bi-fidelity iterative Ensemble Kalman Inversion




**Leverage the accuracy of hi-fidelity model and efficiency of low-fidelity model**

## Prior

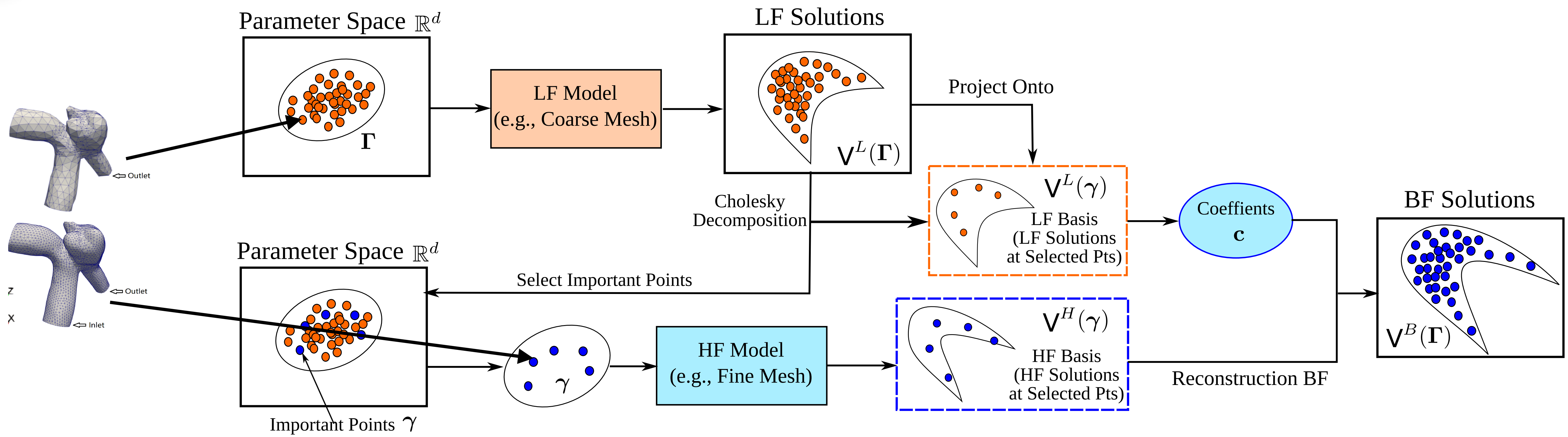
## Forward prediction




**Bi-fidelity strategy**

**Proposed bi-fidelity iterative ensemble Kalman inversion approach**

# Bi-fidelity iterative Ensemble Kalman Inversion




## Empirical error estimation for BF solver

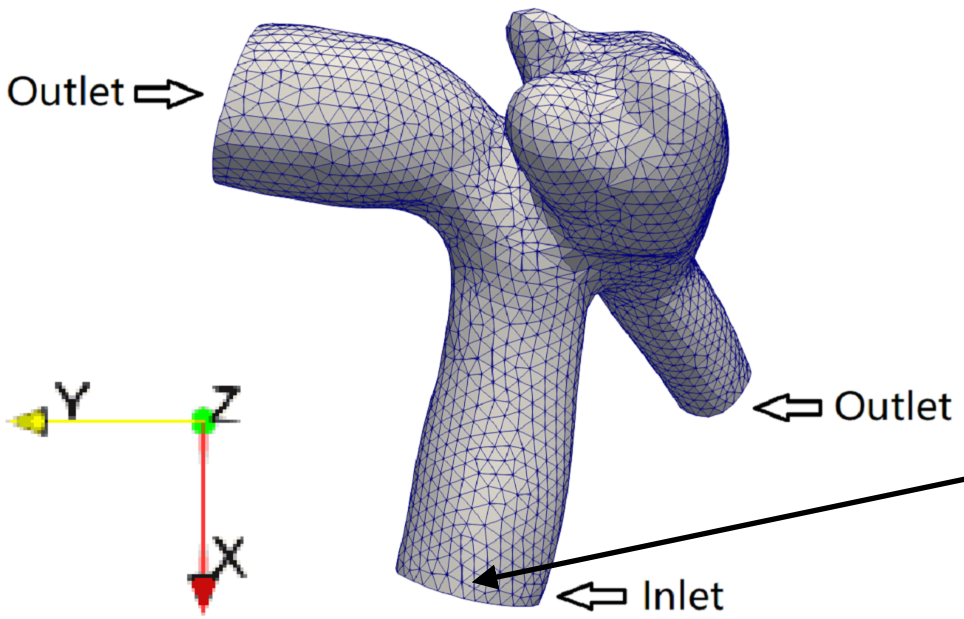
**Theorem 1.** *Given the first $k+1$ pre-selected important points $\gamma^{k+1}$, the relative error between the bi-fidelity solution and the high-fidelity solution can be bounded for any point $\mathbf{z}_* \in \Gamma$ as follows:*

$$\frac{||\mathbf{v}^H(\mathbf{z}_*) - \mathbf{v}^B(\mathbf{z}_*)||}{||\mathbf{v}^H(\mathbf{z}_*)||} \leq \underbrace{\frac{d^H(\mathbf{v}^H(\mathbf{z}_*), \mathbb{U}^H(\gamma^k)))}{||\mathbf{v}^H(\mathbf{z}_*)||}}_{\textit{relative distance}} + \underbrace{\frac{||P_{\mathbb{U}^H(\gamma^k)}\mathbf{v}^H(\mathbf{z}_*) - \mathbf{v}^B(\mathbf{z}_*)||}{||\mathbf{v}^H(\mathbf{z}_*)||}}_{\textit{in-plane error}} \tag{11}$$

$$= \frac{d^H(\mathbf{v}^H(\mathbf{z}_*), \mathbb{U}^H(\gamma^k))}{||\mathbf{v}^H(\mathbf{z}_*)||}\left(1 + \frac{\frac{||P_{\mathbb{U}^H(\gamma^k)}\mathbf{v}^H(\mathbf{z}_*) - \mathbf{v}^B(\mathbf{z}_*)||}{||\mathbf{v}^H(\mathbf{z}_*)||}}{\frac{d^H(\mathbf{v}^H(\mathbf{z}_*), \mathbb{U}^H(\gamma^k)))}{||\mathbf{v}^H(\mathbf{z}_*)||}}\right),$$

H. Gao, X. Zhu, J.X. Wang "**A Bi-fidelity Surrogate Modeling Approach for Uncertainty Propagation in Three-Dimensional Hemodynamic Simulations**", 2019

**Test Case**: Inlet inversion in aneurysm bifurcation




Not known

3D Inlet velocity field

Infer

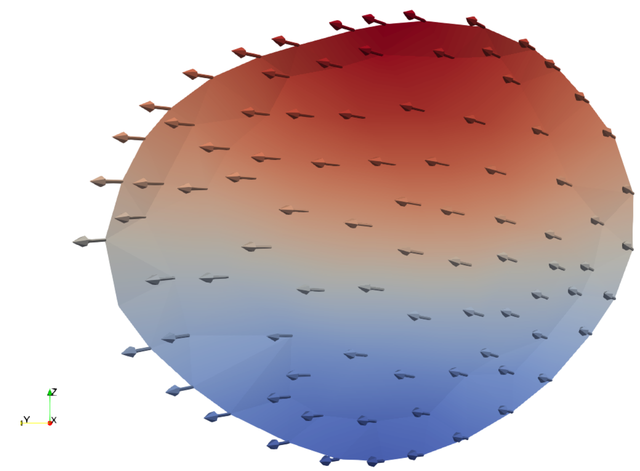
**Sparse, noisy flow measurements**

# Bi-fidelity iterative Ensemble Kalman Inversion

**Test Case**: Inlet inversion in aneurysm bifurcation

Generate synthetic data

Sample out 0.4% flow data, corrupted with 40% noise



Inlet ground truth (Not known)

## Prior distribution of inlet

Gaussian process

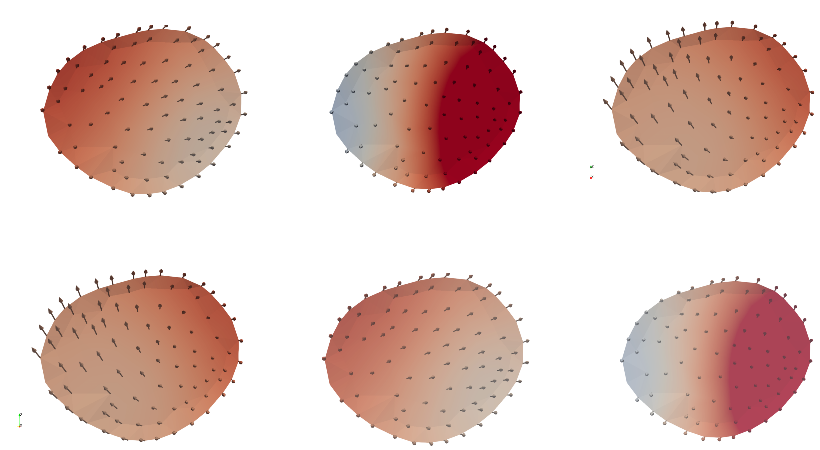
$$f(\mathbf{x}) \sim \mathcal{GP}(0, K(\mathbf{x}, \mathbf{x}')), \;\; K(\mathbf{x}, \mathbf{x}') = \sigma_0^2 \exp(\frac{|\mathbf{x} - \mathbf{x}'|}{2l^2})$$

$$f(\mathbf{x}) = \sum_{i=1}^{n_k \to \infty} \sqrt{\lambda_i} \phi_i(\mathbf{x}) \omega_i$$

$$\mathbf{u}_{in} = \mathbf{u}_{in}^{base} + [f_x(\mathbf{x}), f_y(\mathbf{x}), f_z(\mathbf{x})]$$

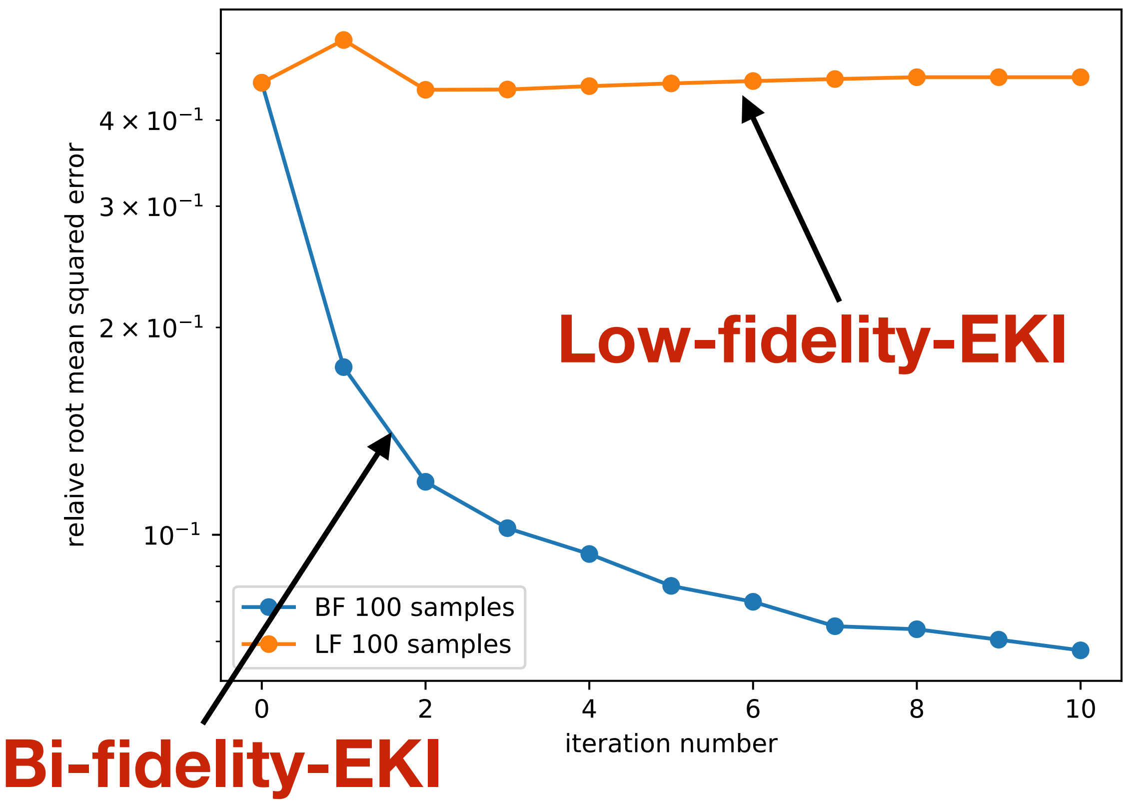
9-dimensional inverse problem

Some prior samples



# Bi-fidelity iterative Ensemble Kalman Inversion

## Convergence history

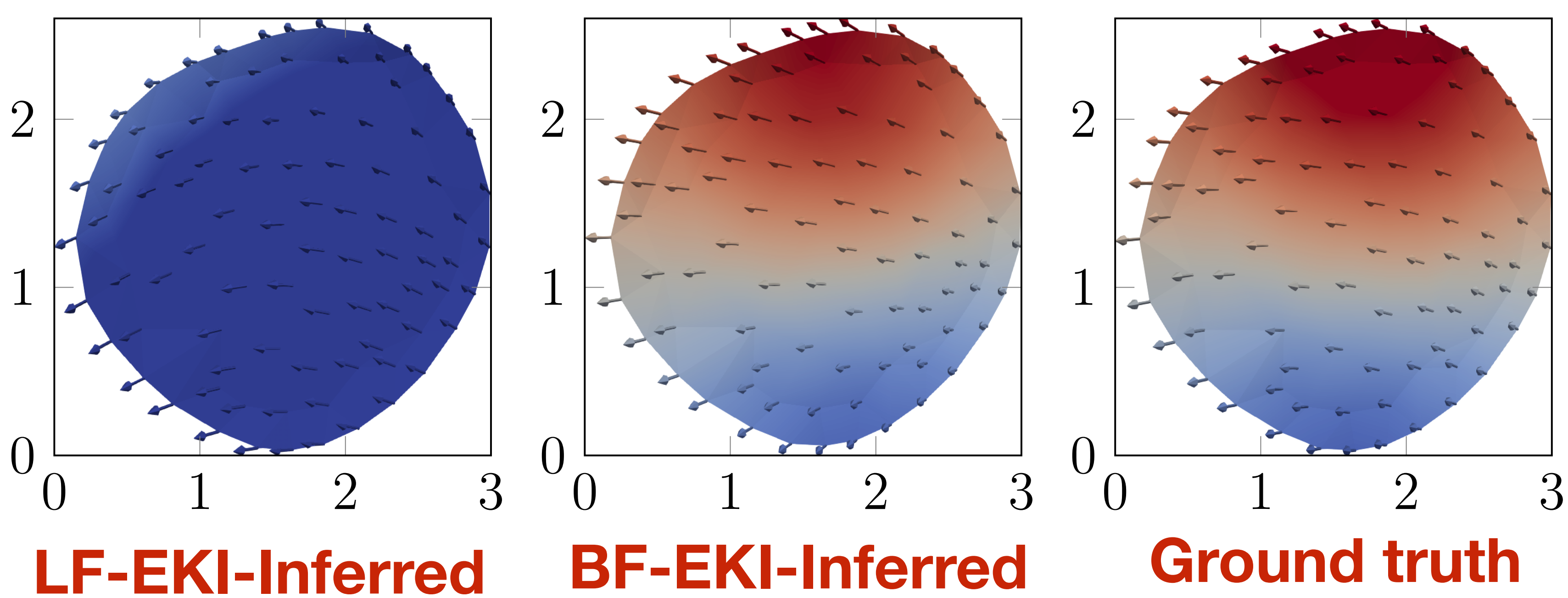



**100 Samples for each iteration for 10 iterations**

**Hi-fidelity-EKI: 30 hours**

**Bi-fidelity-EKI: 0.15 hours (9 mins)!**

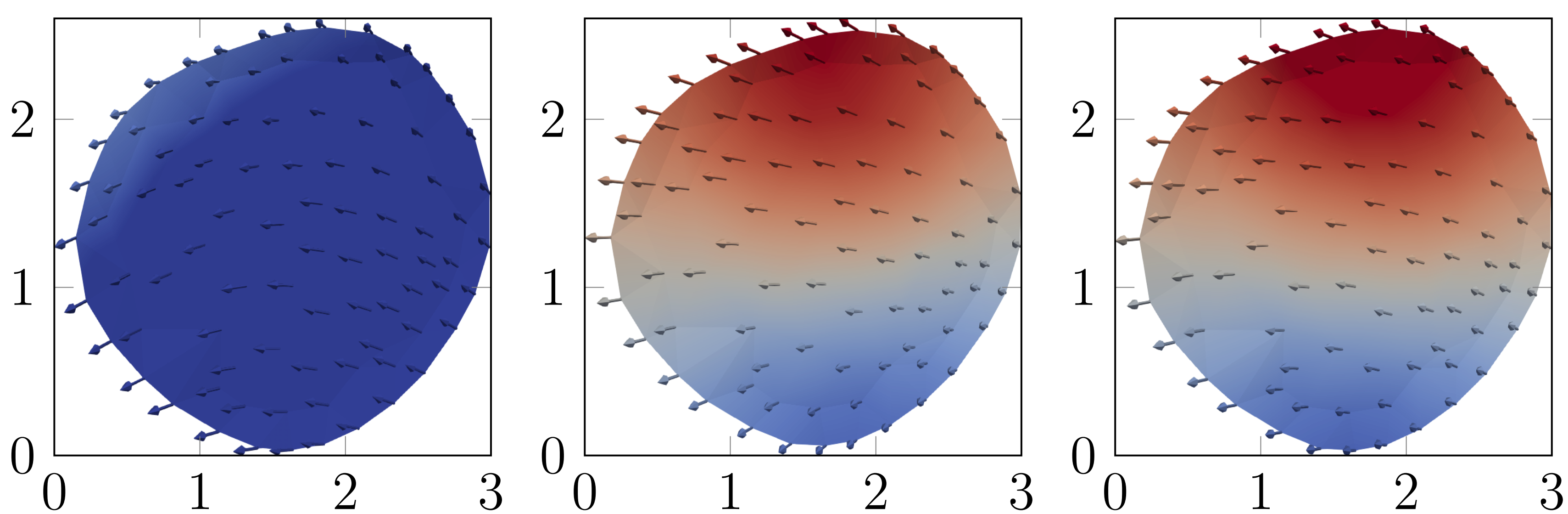
**More than 200 times speedup!**



# Bi-fidelity iterative Ensemble Kalman Inversion

## How do we quantify the uncertainty?

We need to use fully Bayesian method to quantify the uncertainty!




**LF-EKI-Inferred** **BF-EKI-Inferred** **Ground truth**

# MCMC algorithm to quantify the uncertainty

**Fully Bayesian or MCMC methods are computationally expensive; however, in order to validate our method, we did a proof-of-concept case to illustrate how it works:**

1. Simplify the case to an 2D steady simulation of aorta dissection and infer inlet velocity magnitude from observation data from 4D flow MRA

2. From EKI, we obtained an accurate mean U=0.6. Then we assign a prior within interval (0,1) and perform MCMC.

$$P(\theta) = Beta(1.1,1.1)$$

3. With the information from 4D flow MRA on the probing points distributed downstream of the aorta, the final posterior is inferred. Thus, inlet profile is predicted.

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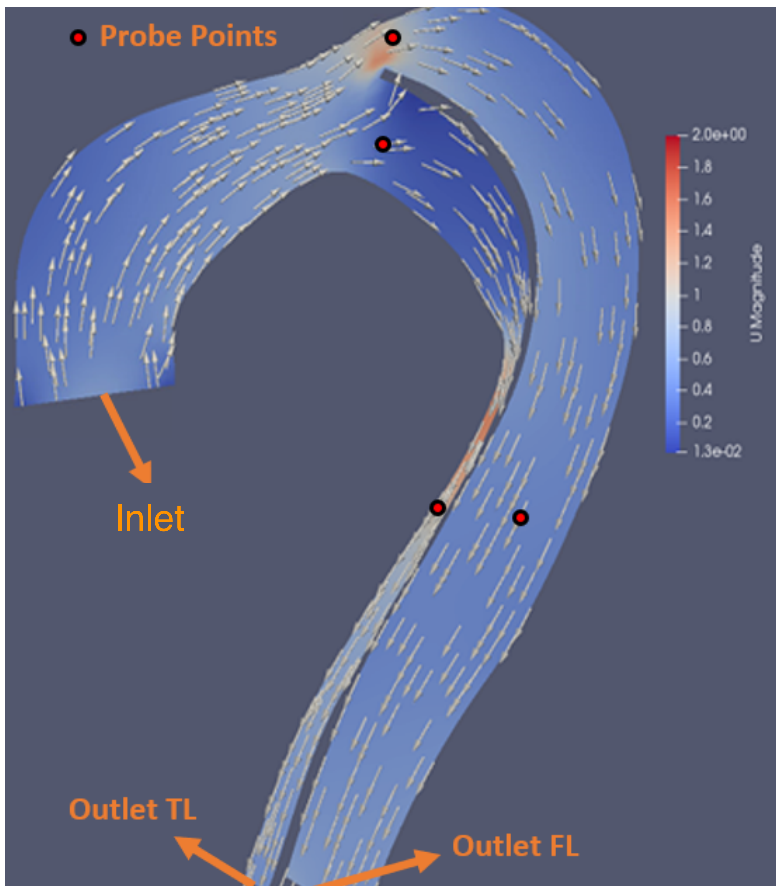
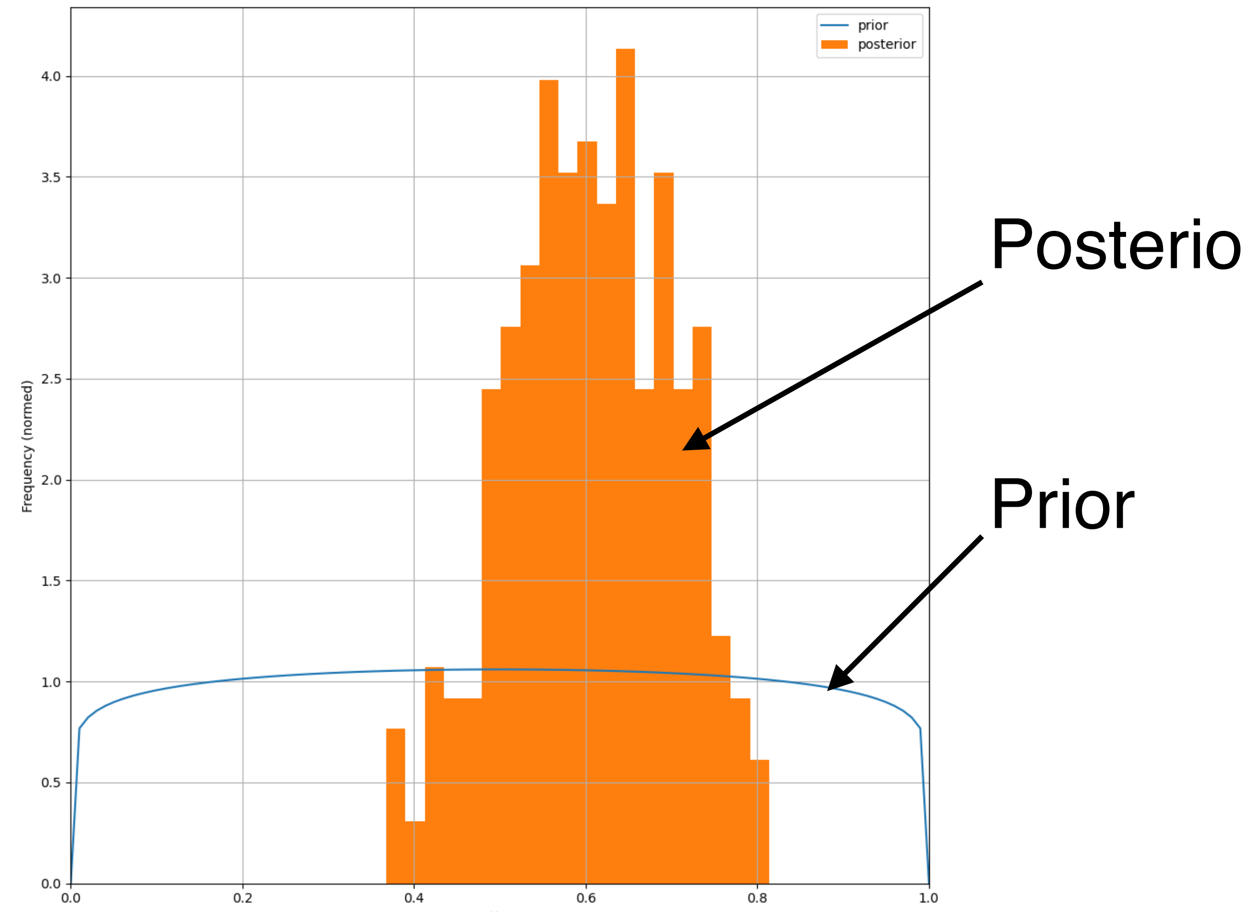
# MCMC algorithm to quantify the uncertainty

**Test Case**: Inlet inversion in aorta dissection

**2D steady simulation cost 1.5 seconds per case**




Probes on a 2D aorta flow filed




Prior vs. Posterior

The MCMC method successfully give the uncertainty of the velocity.

# Conclusion

1. We proposed a Bi-fidelity EKI method and applied it on aneurysm bifurcation inversion problem. The method leverages the accuracy of hi-fidelity model and efficiency of low-fidelity model and yields accurate mean inlet velocity profile with economical computational cost.

2. Besides predicting mean accurately, we have to quantify uncertainty. We adopted MCMC to quantify the velocity magnitude. A aorta dissection case is used for validation and the proposed method gives accurate uncertainty.

3. Future work will apply EKI and fully Bayesian method on 3D simulations to predict mean and uncertainty accurately.

# Related Publications


1. H. Gao, X. Zhu, J.-X. Wang. "**A Bi-fidelity Surrogate Modeling Approach for Uncertainty Propagation in Three-Dimensional Hemodynamic Simulations**". *CMAME*, 2019. (Under revision)[Arxiv]

2. H. Gao, J.-X. Wang. "**A Bi-fidelity Ensemble Kalman Inversion Approach for Inverse Problems in Fluid Simulations**". 2019. (In preparation)

3. J. Wu, J.-X. Wang, S. C. Shadden, "**Improving the Convergence of the Iterative Ensemble Kalman Filter by Resampling**", 2019. [Arxiv]

4. J. Wu, J.-X. Wang, S. C. Shadden, "**Adding constraints to Bayesian inverse problems**", 2019 AAAI Conference on Artificial Intelligence, 2019

# Thank you!