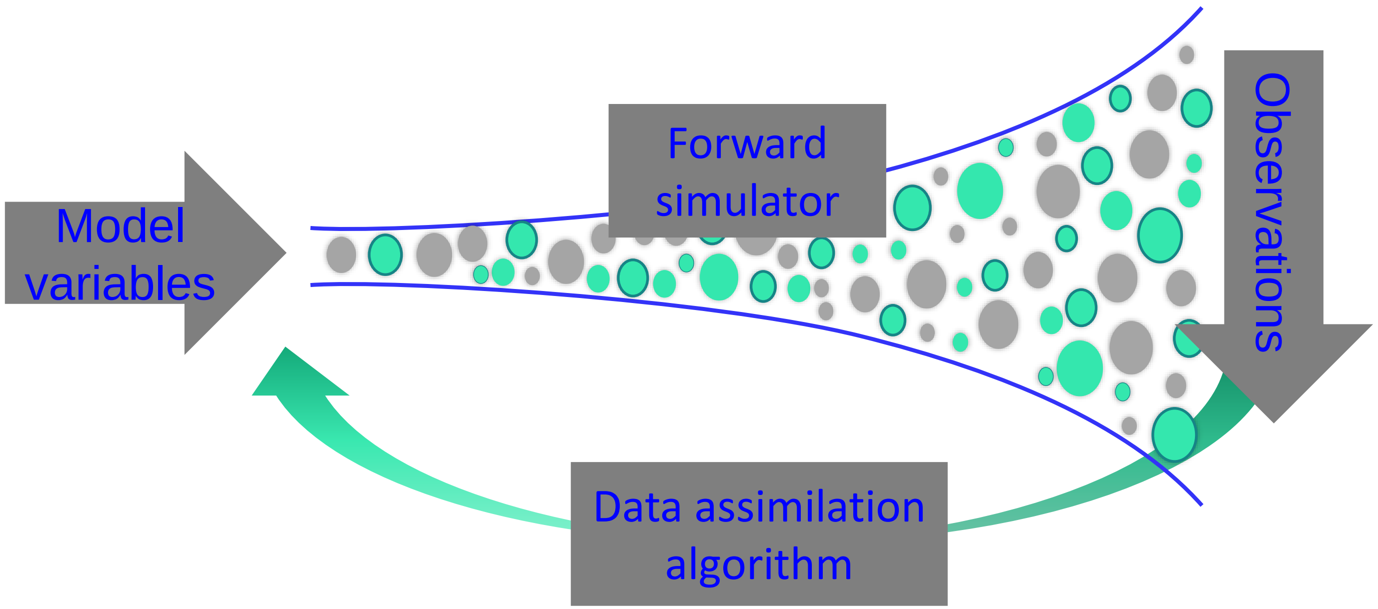
Novel ensemble data assimilation algorithms derived from a class of generalized cost functions

Xiaodong Luo, NORCE

Outline

- An umbrella ensemble data assimilation algorithm
- A class of offspring algorithms with a mixture of regularization terms
- A class of offspring algorithms for data assimilation with soft constraints (DASC)
- Discussion and conclusion

Ensemble data assimilation as a stochastic nonlinear-least-squares (NLS) problem



- Model variables: m
- Linear/nonlinear forward simulator (or observation operators): g
- Observations: d^o

Ensemble data assimilation as a stochastic nonlinear-least-squares (NLS) problem

Stochastic EnKF (SEnKF), ensemble smoother (ES) or iterative ES (IES) can be derived by solving the following stochastic NLS problem*:

$$argmin_{\{m_j^a\}} \frac{1}{N_e} \sum_j L\left(m_j^a | d_j^o, m_j^b, \gamma\right), j = 1, 2, \dots, N_e$$

$$L\left(m^a | d^o, m^b, \gamma\right) = \frac{1}{2}\left(d^o - g(m^a)\right)^T C_d^{-1}\left(d^o - g(m^a)\right) + \frac{\gamma}{2}\left(m^a - m^b\right)^T C_m^{-1}(m^a - m^b)$$

Our focus here is on IES for inverse problems (e.g., reservoir data assimilation problems)

*Luo, X. et al. (2015). Iterative Ensemble Smoother as an Approximate Solution to a Regularized Minimum-Average-Cost Problem: Theory and Applications. *SPE Journal, vol. 20,* 962-982

Ensemble data assimilation as a stochastic nonlinear-least-squares (NLS) problem

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Original IES update formula

$$\mathrm{m}_\mathrm{j}^\mathrm{a} = m_j^\mathrm{b} + S_m S_g \left(S_g S_g^T + \gamma C_d \right)^{-1} \left(d_j^o - g\left(m_j^b\right)\right), j = 1,2,\ldots, N_e$$

$$= m_j^\mathrm{b} + S_m \left(S_g^T C_d^{-1} S_g + \gamma I \right)^{-1} S_g^T C_d^{-1} \left(d_j^o - g\left(m_j^b\right)\right)$$

$$S_m \equiv \frac{1}{\sqrt{N_e - 1}} \left[m_1^b - \bar{m}^b, m_2^b - \bar{m}^b, \ldots, m_{N_e}^b - \bar{m}^b \right]; \ \bar{m}^b = \frac{1}{N_e} \sum_j m_j^b;$$

$$S_g \equiv \frac{1}{\sqrt{N_e - 1}} \left[g\left(m_1^b\right) - g\left(\bar{m}^b\right), g\left(m_2^b\right) - g\left(\bar{m}^b\right), \ldots, g\left(m_{N_e}^b\right) - g\left(\bar{m}^b\right) \right];$$

Ensemble data assimilation beyond NLS problems

$$argmin_{\{m_j^a\}} \frac{1}{N_e} \sum_j L\left(m_j^a | d_j^o, m_j^b, \gamma\right), j = 1, 2, \dots, N_e$$

$$L\left(m^a | d^o, m^b, \gamma\right) = D[\Gamma(d^o) - \Gamma\left(g(m^a)\right)] + \gamma R[\Phi(m^a) - \Phi(m^b)]$$

$\boldsymbol{L\left(m^a | d^o, m^b, \gamma\right)}$ **in general beyond the form of NLS**

Ensemble data assimilation beyond NLS problems

$$L(m^a|d^o, m^b, \gamma) = D[\Gamma(d^o) - \Gamma(g(m^a))] + \gamma R[\Phi(m^a) - \Phi(m^b)]$$

where

- D is a distance metric for the data mismatch term
- Γ is a certain transform operator in the data space
- R is a distance metric for the regularization term
- Φ is another transform operator in the model space

When

- Γ and Φ are identity operator,
- $D(x) = \frac{1}{2} x^T C_d^{-1} x$
- and $R(x) = \frac{1}{2} x^T C_m^{-1} x$, with $C_m = S_m S_m^T$

then we recover the conventional cost function

$$L(m^a|d^o, m^b, \gamma) = \frac{1}{2}\left(d^o - g(m^a)\right)^T C_d^{-1}\left(d^o - g(m^a)\right) + \frac{\gamma}{2}(m^a - m^b)^T C_m^{-1}(m^a - m^b)$$

Ensemble data assimilation beyond NLS problems

Generalized IES (GIES) update formula: the umbrella algorithm*

$$\mathrm{m}_j^{\mathrm{a}} = m_j^{\mathrm{b}} + S_m \left(M_D(\bar{m}^b) + \gamma M_R(m_j^{\mathrm{b}}, \bar{m}^b) \right)^{-1} S_{\Gamma \circ g}^T \, \nabla_D [\Gamma(d^o) - \Gamma(g(m_j^b))]$$

where

- $S_O \equiv \frac{1}{\sqrt{N_e - 1}} [O(m_1^b) - O(\bar{m}^b), O(m_2^b) - O(\bar{m}^b), \ldots, O(m_{N_e}^b) - O(\bar{m}^b)]$ for a generic operator O
- $\nabla_f[x_0] \equiv \frac{\partial f(x)}{x} |_{x_0}$ standing for the gradient of a generic function f evaluated at x_0
- $\nabla_f^2[x_0] = \left(\frac{\partial^2 f}{\partial x^2}\right)^T |_{x_0}$ for the Hessian of f evaluated at x_0
- $M_D(\bar{m}^b) \equiv S_{\Gamma\circ g}^T \nabla_D^2 [\Gamma(d^o) - \Gamma(g(\bar{m}^b))] \, S_{\Gamma\circ g}$, with $\Gamma \circ g(x) \equiv \Gamma(g(x))$
- $M_R(m_j^{\mathrm{b}}, \bar{m}^b) \equiv S_\Phi^T \nabla_R^2 [\Phi(\bar{m}^b) - \Phi(m_j^{\mathrm{b}})] \, S_\Phi$

*Luo, X. (2021). Novel iterative ensemble smoothers derived from a class of generalized cost functions. *Computational Geosciences*, *25*(3), 1159-1189.

Correspondence between the update formulae of IES and GIES

IES	GIES
$m_j^a = m_j^b + S_m\left(S_g^T C_d^{-1} S_g + \gamma I\right)^{-1} S_g^T C_d^{-1}\left(d^o - g\left(m_j^b\right)\right)$	$m_j^a = m_j^b + S_m\left(M_D(\bar{m}^b) + \gamma M_R\left(m_j^b, \bar{m}^b\right)\right)^{-1} S_{\Gamma\circ g}^T \nabla_D[\Gamma(d^o) - \Gamma(g(m_j^b))]$

IES	GIES	Comment
$C_d^{-1}\left(d^o - g\left(m_j^b\right)\right)$	$\nabla_D[\Gamma(d^o) - \Gamma(g(m_j^b))]$	GIES => IES if Γ = identity, $D(x) = \frac{1}{2}x^T C_d^{-1} x$
S_m/S_g	$S_\Phi/S_{\Gamma\circ g}$	GIES => IES if Φ/Γ = Identity,
C_m^{-1}/C_d^{-1}	$\nabla_R^2[\Phi(\bar{m}^b) - \Phi(m_j^b)]/\nabla_D^2[\Gamma(d^o) - \Gamma(g(\bar{m}^b))]$	GIES => IES if $R(x) = \frac{1}{2}x^T C_m^{-1} x / D(x) = \frac{1}{2}x^T C_d^{-1} x$
$S_g^T C_d^{-1} S_g$	$M_D(\bar{m}^b) \equiv S_{\Gamma\circ g}^T \nabla_D^2[\Gamma(d^o) - \Gamma(g(\bar{m}^b))]\, S_{\Gamma\circ g}$	GIES => IES if Γ = identity, $D(x) = \frac{1}{2}x^T C_d^{-1} x$
I	$M_R(m_j^b, \bar{m}^b) \equiv S_\Phi^T \nabla_R^2[\Phi(\bar{m}^b) - \Phi(m_j^b)]\, S_\Phi$	GIES => IES if Φ = identity, $R(x) = \frac{1}{2}x^T C_m^{-1} x$, with $C_m = S_m S_m^T$

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ℓ_p^q-GIES as a class of offspring algorithms

$$L\left(m^a|d^o,m^b,\gamma\right)=\frac{1}{2}\left(d^o-g(m^a)\right)^T C_d^{-1}\left(d^o-g(m^a)\right)+\frac{\gamma}{2}R[\Phi(m^a)-\Phi(m^b)]$$

$$R[\Phi(m^a)-\Phi(m^b)]=\sum_{i=1}^{K} w_i\,\|\mathbf{B}_i(\Phi_i(m^a)-\Phi_i(m^b))\|_{p_i}^{q_i},\ p_i/q_i\in R_+$$

- $w_i/\mathbf{B}_i/\ \Phi_i$: mixture coefficient/weight matrix/transform operator for the i-th regularization term
- For $\mathbf{B}\in \boldsymbol{R}^{m_b\times m_x}$, $\boldsymbol{x}\in R^{m_x}$, the ℓ_p^q metric of the vector $\mathbf{B}\boldsymbol{x}\in R^{m_b}$ defined as

$$\|\mathbf{B}\boldsymbol{x}\|_p^q\equiv\left(\sum_{e=1}^{m_b}|(\mathbf{B}\boldsymbol{x})_e|^p\right)^{\frac{q}{p}}$$

$(\mathbf{B}\boldsymbol{x})_e=\sum_{f=1}^{m_x} B_{e,f}\,x_f$ the e-th element of $\mathbf{B}\boldsymbol{x}$, $B_{e,f}/x_f$ elements of $\mathbf{B}/\boldsymbol{x}$

ℓ_p^q-GIES as a class of offspring algorithms

Update formula of ℓ_p^q-GIES*

$$\mathrm{m_j^a} = m_j^{\mathrm{b}} + S_m\left(S_g^T C_d^{-1} S_g + \gamma M_R\left(m_j^{\mathrm{b}}, \bar{m}^b\right)\right)^{-1} S_g^T C_d^{-1}\left(d^o - g\left(m_j^b\right)\right)$$

$R[\Phi(m^a) - \Phi(m^b)] = \frac{1}{2}\sum_{i=1}^{K} w_i \lVert \mathbf{B}_i(\Phi_i(m^a) - \Phi_i(m^b)) \rVert_{p_i}^{q_i}$	$M_R\left(m_j^{\mathrm{b}}, \bar{m}^b\right) = \frac{1}{2}\sum_{i=1}^{K} w_i S_{\Phi_i}^T \nabla^2_{\lVert \mathbf{B}_i(\cdot) \rVert_{p_i}^{q_i}}\left[\Phi_i(\bar{m}^b) - \Phi_i(m_j^{\mathrm{b}})\right] S_{\Phi_i}$ $\nabla^2_{\lVert \mathbf{B}_i(\cdot) \rVert_{p_i}^{q_i}}\left[\Phi_i(\bar{m}^b) - \Phi_i(m_j^{\mathrm{b}})\right] =$ $q_i(q_i\text{-}p_i) \lVert \mathbf{B}_i(\Phi_i(\bar{m}^b) - \Phi_i(m^b)) \rVert_{p_i}^{q_i - 2p_i}\ \mathbf{B}_i^{\mathrm{T}} \boldsymbol{a}_i \boldsymbol{a}_i^{\mathrm{T}} \mathbf{B}_i\ +$ $q_i(p_i\text{-}1) \lVert \mathbf{B}_i(\Phi_i(\bar{m}^b) - \Phi_i(m^b)) \rVert_{p_i}^{q_i - p_i}\ (\mathbf{B}_i \odot \boldsymbol{C}_i)^{\mathrm{T}} (\mathbf{B}_i \odot \boldsymbol{C}_i);$ $\boldsymbol{a}_i \equiv \left\lvert \mathbf{B}_i\left(\Phi_i(\bar{m}^b) - \Phi_i(m^b)\right) \right\rvert^{\dot{\wedge}\,(\boldsymbol{p_i - 2})} \odot \left(\mathbf{B}_i\left(\Phi_i(\bar{m}^b) - \Phi_i(m^b)\right)\right)$ $\boldsymbol{C}_i \equiv \left\lvert \mathbf{B}_i\left(\Phi_i(\bar{m}^b) - \Phi_i(m^b)\right) \right\rvert^{\dot{\wedge}\,(\boldsymbol{p_i/2 - 1})}\ \mathbf{1}^{\mathrm{T}}$
Notations	$\odot$: Schur (or element-wise) product; $\dot{\wedge}$: Raising all elements of a vector to a certain power

*Luo, X. (2021). Novel iterative ensemble smoothers derived from a class of generalized cost functions. *Computational Geosciences*, *25*(3), 1159-1189.

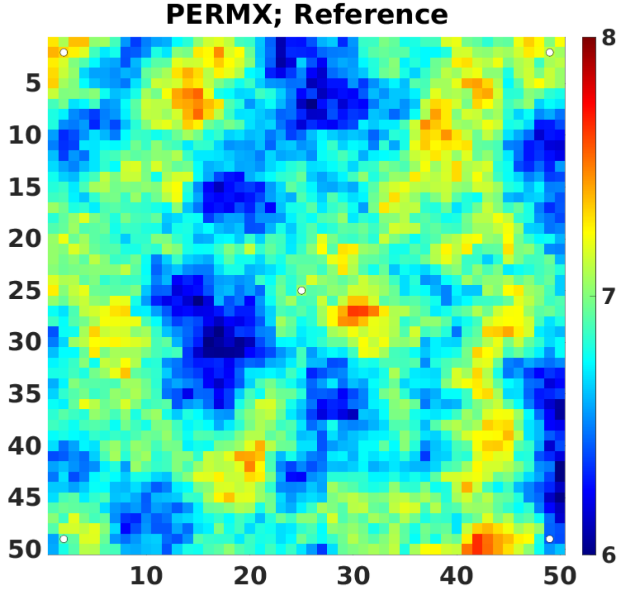
ℓ_p^q-GIES as a class of offspring algorithms

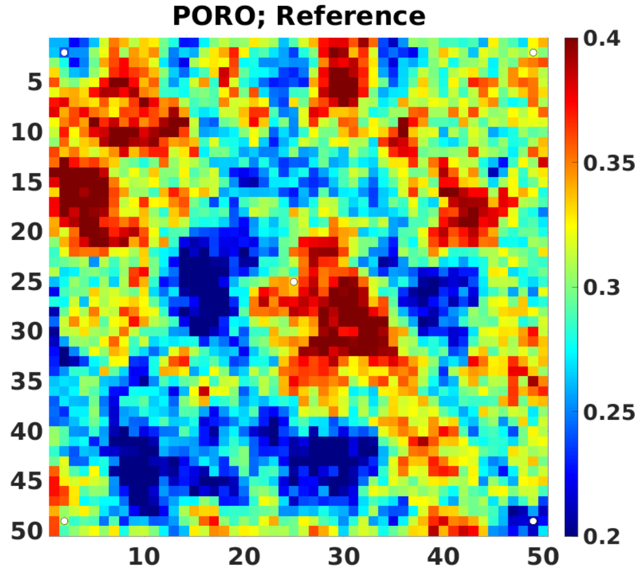
$$R[\Phi(m^a) - \Phi(m^b)] = \sum_{i=1}^{K} w_i \, \|\mathbf{B}_i(\Phi_i(m^a) - \Phi_i(m^b))\|_{p_i}^{q_i}$$

- When $p = q = 2, K = 1,$ the ℓ_2^2-GIES algorithm is reduced to the original IES in Luo et al.*
- In general, infinitely many choices for the (p, q) pair (p, q not necessarily being integers), leading to ℓ_p^q-GIES algorithms beyond the form of nonlinear-least-squares in general
- Also many choices for $\mathbf{B}_i/\Phi_i$

*Luo, X. et al. (2015). Iterative Ensemble Smoother as an Approximate Solution to a Regularized Minimum-Average-Cost Problem: Theory and Applications. *SPE Journal, vol. 20,* 962-982

Applications of ℓ_p^q-GIES: Case study 2





Model size (gridblock)	50 x 50
Phases	Oil, gas and water
Wells	4 producers + 1 injector
Data for history matching	BHP, WWPR, WGPR and WOPR, from Day 1 – Day 1500
Parameters to estimate	Permeability (PERM) and porosity (PORO) on all gridblocks
History matching algorithm	7 ℓ_p^q-GIES (including the original IES), with 100 ensemble members + correlation based adaptive localization, and 10 iteration steps

Applications of ℓ_p^q-GIES: Case study 2

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Results
(more information available
in the paper*)

Table 4 Performance of ℓ_p^q-GIES algorithms in terms of RMSE, which are evaluated with respect to the ensembles of reservoir models at the final iteration steps

Rank	Binary code	History-matching data mismatch (mean ± STD)	RMSE of PORO (mean ± STD)	RMSE of PERMX (mean ± STD)	Weights $(\alpha_1, \alpha_2, \alpha_3)$
1	001	1191.5353 ± 1455.8489	0.0619 ± 0.0035	0.4045 ± 0.025	(0, 0, 1)
2	010	502.441 ± 96.276	0.0637 ± 0.0033	0.4156 ± 0.0235	(0, 1, 0)
3	111	492.3271 ± 86.3171	0.0637 ± 0.0033	0.4194 ± 0.024	(0.4, 0.4, 0.2)
4	110	488.5942 ± 86.6626	0.0635 ± 0.0033	0.4202 ± 0.0239	(0.5, 0.5, 0)
5	011	548.0362 ± 314.1955	0.0636 ± 0.0033	0.4208 ± 0.0242	(0, 0.5, 0.5)
6	100	501.1523 ± 94.3388	0.0635 ± 0.0033	0.4211 ± 0.0244	(1, 0, 0)
7	101	498.7347 ± 92.2996	0.0637 ± 0.0033	0.4238 ± 0.0246	(0.5, 0, 0.5)

The ℓ_p^q-GIES algorithms are listed in an ascending order of mean RMSE values. In particular, performance of the ℓ_p^q-GIES algorithm corresponding to the original IES is highlighted (in red)

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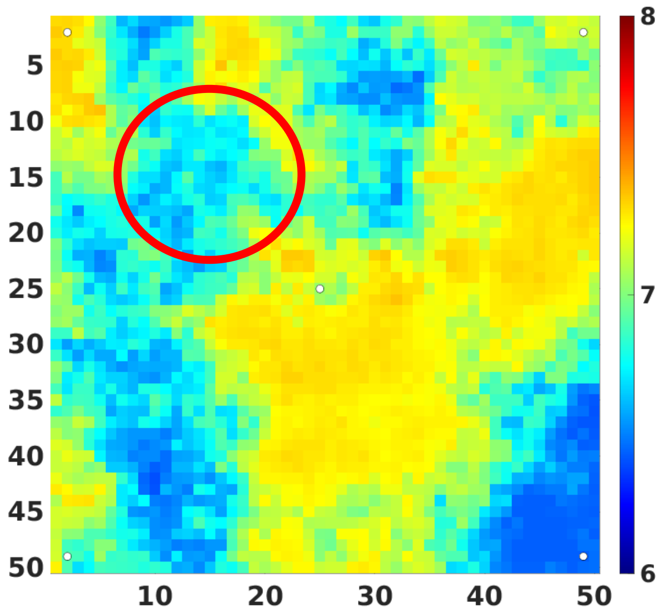
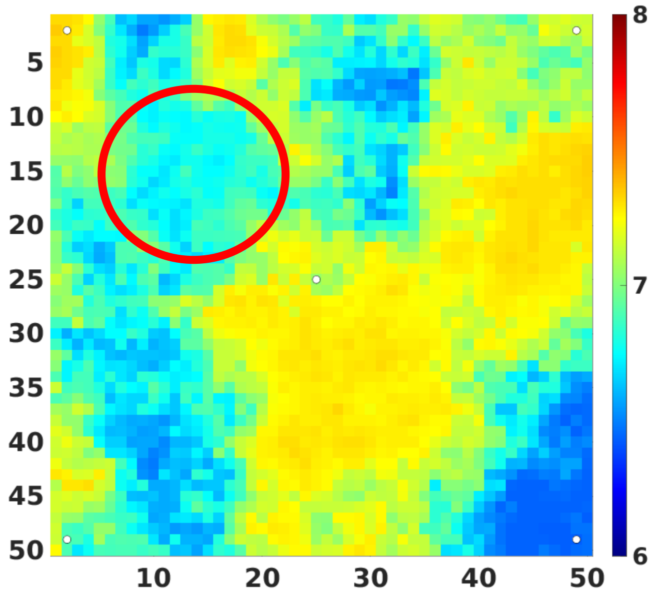
Applications of ℓ_p^q-GIES: Case study 2

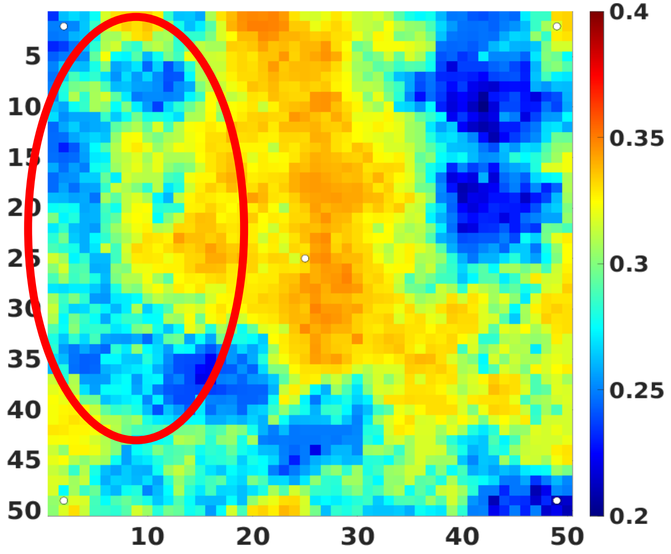


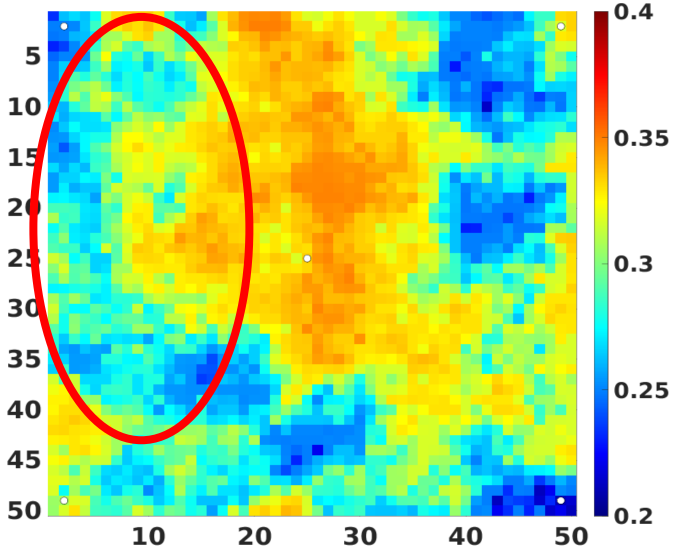
PERM estimated by the ℓ_2^2-GIES (the original IES)



PERM estimated by the ℓ_1^2-GIES (achieving the best results in this case study)

Applications of ℓ_p^q-GIES: Case study 2





PORO estimated by the ℓ_2^2-GIES (the original IES)

PORO estimated by the ℓ_1^2-GIES (achieving the best results in this case study)

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Constrained GIES (C-GIES) for data assimilation with soft constraints (DASC)

Available sources of information in a DASC problem, :

- Original observation system: $\boldsymbol{d}^{sim} = \boldsymbol{g}(\boldsymbol{m})$
- Equality constraint system: $\boldsymbol{f}_{eq}(\boldsymbol{m}) = \boldsymbol{0}$
- Inequality constraint system: $\boldsymbol{h}_{in}(\boldsymbol{m}) \leq \boldsymbol{0}$

Constrained GIES (C-GIES) for data assimilation with soft constraints (DASC)

$$L\left(m^a | d^o, m^b\right) = D\left[\Gamma(d^o) - \Gamma\left(g(m^a)\right)\right] + \frac{\gamma}{2}\left(m - m^b\right)^T C_m^{-1}\left(m - m^b\right)$$

$$D\left[\Gamma(d^o) - \Gamma\left(g(m^a)\right)\right] = \frac{1}{2}\left(d^o - g(m^a)\right)^T C_d^{-1}\left(d^o - g(m^a)\right) + \alpha\, D_{eq}\left(0 - f_{eq}(m^a)\right) + \beta\, D_{in}\left(0 - h_{in}(m^a)\right)$$

Constrained GIES (C-GIES) for data assimilation with soft constraints (DASC)

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Update formula of C-GIES*

$$\mathrm{m_j^a} = m_j^{\mathrm{b}} + K\left(S_g^T C_d^{-1}\left(d^o - g\left(m_j^{\mathrm{b}}\right)\right) + \alpha S_{\mathrm{f_{eq}}}^T \nabla_{D_{eq}}[0 - f_{eq}(m_j^{\mathrm{b}})] + \beta S_{h_{in}}^T \nabla_{D_{in}}[0 - h_{in}(m_j^{\mathrm{b}})]\right)$$

$$K \equiv S_m\left(S_{\mathrm{g}}^T\, C_d^{-1}\, S_{\mathrm{g}} + \alpha S_{\mathrm{f_{eq}}}^T \nabla_{D_{eq}}^2[0 - f_{eq}(\bar{m}^b)] S_{\mathrm{f_{eq}}} + \beta S_{h_{in}}^T \nabla_{D_{in}}^2[0 - h_{in}(\bar{m}^b)] S_{\mathrm{h_{in}}} + \gamma I\right)^{-1}$$

Red: impact of equality constraints on model update

Green: impact of inequality constraints on model update

$\alpha = \beta = 0 \Rightarrow$ original IES algorithm

*Luo, X., Cruz, W. (2021). Data assimilation with soft constraints (DASC) through a generalized iterative ensemble smoother. Submitted for review

Constrained GIES (C-GIES) for data assimilation with soft constraints (DASC)

$$\mathrm{m}_{\mathrm{j}}^{\mathrm{a}} = m_j^{\mathrm{b}} + K\left(S_g^T C_d^{-1}\left(d^o - g\left(m_j^{\mathrm{b}}\right)\right) + \alpha S_{\mathrm{f_{eq}}}^T \nabla_{D_{eq}}[0 - f_{eq}\left(m_j^{\mathrm{b}}\right)] + \beta S_{h_{in}}^T \nabla_{D_{in}}[0 - h_{in}\left(m_j^{\mathrm{b}}\right)]\right)$$

Leveraging efficient solutions to the following two problems*:

- Localization in the presence of constraints
- High dimensionality of the constraint system

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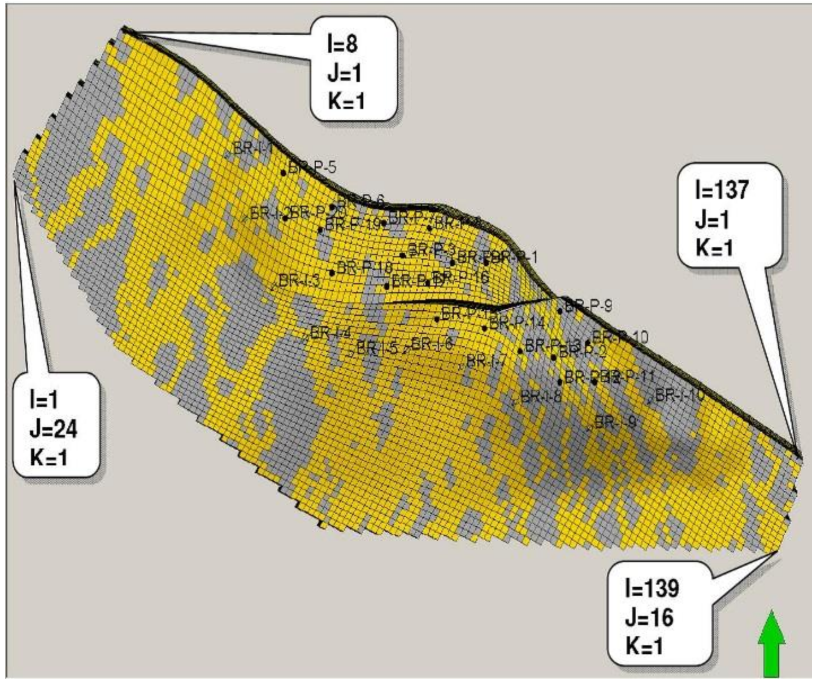
Numerical example 2: 3D Brugge field



Grid geometry of the Brugge field

Experimental settings	
Model size	139x48x9, with 44550 out of 60048 being active gridcells
Parameters to estimate	PORO, PERMX, PERMY, PERMZ. Total number is 4x44550 = 178,200
Production data (~10 yrs)	BHP, OPR, WCT. Total number is 1400
Constraint system	Upper and lower bounds for each parameter. Dimension of the constraint system = 2 x 178,200 = 356,4000
History matching algorithm	C-GIES + correlation-based adaptive localization, 10 iteration steps

Numerical example 2: 3D Brugge field

Table 3: Performance of the two history-matching algorithms in the Brugge case study. The performance is measured in terms of RMSE (mean ± STD), which are computed using the ensembles of reservoir models at the first and final iteration steps. Other quantities reported here include data mismatch during history matching, and the value of barrier function (in the form of mean ± STD), with respect to both the initial and final ensembles. For RMSE, the values are calculated with respect to PERMX, PERMY, PERMZ (in the scale of natural logarithm), PORO, and the combination of all these variables, respectively.

	Initial ensemble	O-IES	C-GIES-IN
Data mismatch	$3.6232 \times 10^9 \pm 1.4900 \times 10^{10}$	$(3.9616 \pm 2.9947) \times 10^7$	$(7.0091 \pm 5.5507) \times 10^6$
Value of barrier function	$-3.4172 \times 10^5 \pm 6.6936 \times 10^3$	$-3.4217 \times 10^5 \pm 5.9683 \times 10^3$	$-3.4258 \times 10^5 \pm 3.9202 \times 10^3$
RMSE (PERMX)	1.6585 ± 0.3827	1.4167 ± 0.2545	1.4119 ± 0.2284
RMSE (PERMY)	1.6612 ± 0.3794	1.4198 ± 0.2515	1.4133 ± 0.2244
RMSE (PERMZ)	2.0077 ± 0.4096	1.8054 ± 0.3101	1.7636 ± 0.2916
RMSE (PORO)	0.0302 ± 0.0033	0.0280 ± 0.0025	0.0285 ± 0.0028
RMSE (all together)	1.5450 ± 0.3362	1.3498 ± 0.2344	1.3327 ± 0.2103

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Discussion and conclusion

- GIES as an umbrella algorithm, able to derive infinitely many new IES
  - ℓ_p^q-GIES
  - C-GIES
  - Likely more
- Applicable to large scale problems
- Remaining open problems
  - Optimal choices of weight coefficients (e.g., α, β)
  - Optimal choices of the cost functional $D[\Gamma(d^o) - \Gamma(g(m^a))] + \gamma R[\Phi(m^a) - \Phi(m^b)]$ in various problems

Acknowledgements / Thank You / Questions

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Backup slides

Applications of ℓ_p^q-GIES: Case study 1

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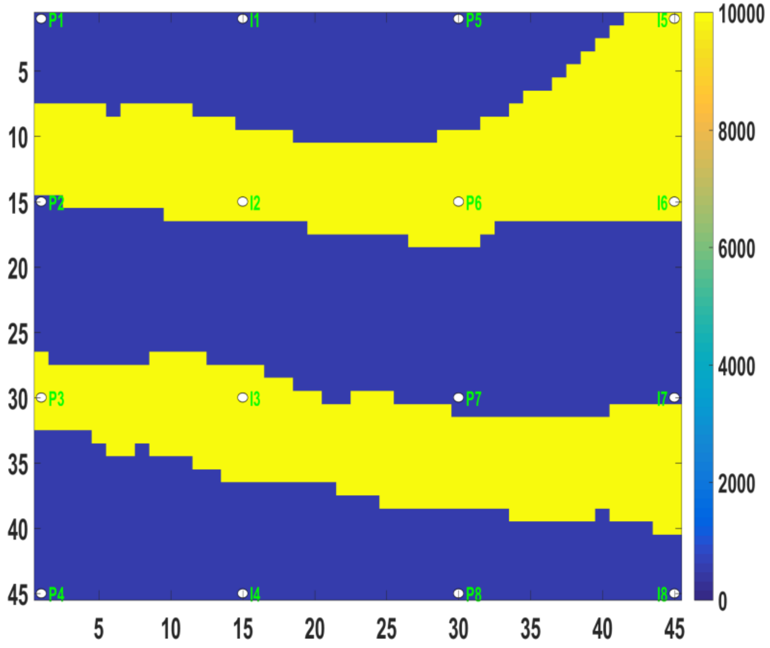
Reference permeability field



Model size (gridblock)	45 x 45
Phases	Oil and water
Wells	8 producers (P1-P8) and 8 injectors (I1-I8)
Data for history matching	BHP from injectors + OPR and WPR from producers, from Day 1 – Day 1900
Data for cross-validation	Forecast BHP from injectors + forecast OPR and WPR from producers, from Day 1901 – Day 3800
Parameters to estimate	Permeability on all gridblocks
History matching algorithm	31 ℓ_p^q-GIES (including the original IES), with 100 ensemble members + correlation based adaptive localization, and 50 iteration steps

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Table 2: Performance of ℓ_p^q-GIES algorithms in terms of data mismatch values during the history matching and forecast periods, which are evaluated with respect to the ensembles of reservoir models at the final iteration steps. The ℓ_p^q-GIES algorithms are listed in an ascending order of mean values of forecast data mismatch. In particular, the ℓ_p^q-GIES algorithm corresponding to the original IES is highlighted (in red).

Rank	Binary code	History-matching data mismatch (mean ± STD)	Forecast data mismatch (mean ± STD)	Weights $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$
1	11000	1359.6202 ± 357.7710	2695.0909 ± 511.6914	(0.8,0.2,0,0,0)
2	10011	1274.8102 ± 352.7197	2742.4803 ± 528.4256	(0.4,0,0,0.3,0.3)
3	01001	1059.2603 ± 407.6890	2747.4506 ± 744.9570	(0,0.5,0,0,0.5)
4	10010	1628.4537 ± 426.3572	2782.8461 ± 580.7321	(0.8,0,0,0.2,0)
5	01111	1307.1619 ± 560.2085	2827.8018 ± 635.4243	(0,0.25,0.25,0.25,0.25)
6	01011	1290.0505 ± 340.6488	2852.5673 ± 601.0067	(0,0.2,0,0.4,0.4)
7	10001	1375.3988 ± 404.3301	2889.5725 ± 440.7436	(0.8,0,0,0,0.2)
8	11001	1639.6244 ± 397.6959	2967.0104 ± 680.8220	(0.4,0.3,0,0,0.3)
9	01010	1247.8989 ± 308.4266	3030.2979 ± 647.2721	(0,0.5,0,0.5,0)
10	11100	1331.0531 ± 456.9900	3090.6186 ± 651.3856	(0.4,0.3,0.3,0,0)
11	01101	1431.2121 ± 899.7220	3114.9221 ± 1597.5630	(0,0.4,0.4,0,0.2)
12	01110	1309.4299 ± 474.6920	3232.0429 ± 866.5564	(0,0.4,0.4,0.2,0)
13	10110	1472.9386 ± 662.7470	3290.7245 ± 1084.1258	(0.4,0,0.3,0.3,0)
14	10111	1469.2809 ± 672.3853	3341.4622 ± 1150.9453	(0.25,0,0.25,0.25,0.25)
15	10000	2296.4291 ± 1149.3603	3372.0041 ± 994.2162	(1,0,0,0,0)
16	11010	1638.5585 ± 477.1788	3374.6603 ± 705.6481	(0.4,0.3,0,0.3,0)
17	01100	1472.6951 ± 866.7636	3383.6922 ± 1929.6565	(0,0.5,0.5,0,0)
18	10100	2153.5187 ± 1051.0361	3445.6954 ± 1083.8990	(0.8,0,0.2,0,0)
19	11011	1436.6160 ± 428.4257	3450.4318 ± 745.0217	(0.25,0.25,0,0.25,0.25)
20	10101	1618.9679 ± 491.3947	3456.4395 ± 1166.0237	(0.4,0,0.3,0,0.3)
21	01000	1656.5239 ± 682.2189	3492.2594 ± 1555.8071	(0,1,0,0,0)
22	11101	1127.7861 ± 432.7342	3590.6883 ± 895.5180	(0.25,0.25,0.25,0,0.25)
23	11111	1346.4950 ± 603.9574	3941.5423 ± 868.1645	(0.2,0.2,0.2,0.2,0.2)
24	11110	1266.6071 ± 550.7372	4045.9208 ± 913.8584	(0.25,0.25,0.25,0.25,0)
25	00011	5793.5819 ± 2251.2925	7594.0385 ± 3778.9256	(0,0,0,0.5,0.5)
26	00010	5793.5842 ± 2251.2988	7594.0438 ± 3778.9230	(0,0,0,1,0)
27	00100	5793.5850 ± 2251.3011	7594.0499 ± 3778.9334	(0,0,1,0,0)
28	00001	5793.5853 ± 2251.2755	7594.0512 ± 3778.9744	(0,0,0,0,1)
29	00101	5793.5856 ± 2251.2960	7594.0514 ± 3778.9774	(0,0,0.5,0,0.5)
30	00110	5793.5849 ± 2251.2972	7594.0569 ± 3778.9778	(0,0,0.5,0.5,0)
31	00111	5793.5783 ± 2251.2818	7594.0617 ± 3778.9620	(0,0,0.2,0.4,0.4)

Results
(more information
available
in the paper*)

*Luo, X. (2021). Novel iterative ensemble smoothers derived from a class of generalized cost functions. *Computational Geosciences*, *25*(3), 1159-1189.

Application of ℓ_p^q-GIES: Case study 1

Adopting ℓ_p^q-GIES algorithms

$$L\left(m^a|d^o,m^b,\gamma\right)=\frac{1}{2}\left(d^o-g(m^a)\right)^T C_d^{-1}\left(d^o-g(m^a)\right)+\gamma R[\Phi(m^a)-\Phi(m^b)]$$

with R consisting of 5 individual terms with the ℓ_2^2 or ℓ_1^2 metric

$$2R[\Phi(m^a)-\Phi(m^b)]=w_1\|\mathbf{B}_1\left(m^a-m^b\right)\|_2^2+w_2\left\|TV(m^a)-\mathrm{TV}\left(m^b\right)\right\|_2^2+w_3\left\|TV(m^a)-\mathrm{TV}\left(m^b\right)\right\|_1^2 + w_4\left\|IE_{hist}(m^a)-IE_{hist}\left(m^b\right)\right\|_2^2+w_5\left\|IE_{hist}(m^a)-IE_{hist}\left(m^b\right)\right\|_1^2$$

$\mathbf{B}_1^{\mathrm{T}}\mathbf{B}_1=(S_m S_m^T)^{-1}$, and in effect, $\mathbf{B}_i$ *all equal to identity matrices for* $i=2,3,4,5$

TV: operator computing the first-order total variation (TV) of a reservoir model

IE_{hist}: operator computing the information entropy (IE) of the histogram of a reservoir model

Application of ℓ_p^q-GIES: Case study 1

$$2R[\Phi(m^a) - \Phi(m^b)] = w_1\|\mathbf{B}_1(m^a - m^b)\|_2^2 + w_2\|TV(m^a) - \mathrm{TV}(m^b)\|_2^2 + w_3\|TV(m^a) - \mathrm{TV}(m^b)\|_1^2 + w_4\|IE_{hist}(m^a) - IE_{hist}(m^b)\|_2^2 + w_5\|IE_{hist}(m^a) - IE_{hist}(m^b)\|_1^2$$

- When w_1=*1*, $w_i = 0, i = 2,3,4,5$, recovering the original IES
- 5-bit binary encoding system $(e_1 e_2 e_3 e_4 e_5), e_i \in \{0,1\}, i = 1,2,3,4,5$, used to refer the resulting ℓ_p^q-GIES algorithms. If $w_i = 0$, $e_i = 0$; otherwise, $e_i = 1$. Example: the original IES encoded as 10000
- This leads to 31 ℓ_p^q-GIES algorithms in total for performance comparison, excluding the one with the code 00000 (no regularization)
- Data mismatch during the forecast period as the performance measure

Constrained GIES (C-GIES) for data assimilation with soft constraints (DASC)

Inequality constraint system with barrier function (pushing away from the boundary)

$$\mathcal{D}_{in}\left(\mathbf{x}\right) = -\left(\log\left(\mathbf{x}+\mathbf{a}\right)\right)^{T}\mathbf{1}_{len(\mathbf{x})}.$$

$$\boldsymbol{\nabla}_{\mathcal{D}_{in}}\left[\mathbf{x}\right] = -\mathbf{1}_{len(\mathbf{x})}./(\mathbf{x}+\mathbf{a});$$

$$\boldsymbol{\nabla}^{2}_{\mathcal{D}_{in}}\left[\mathbf{x}\right] = \mathrm{diag}\left(\left(\mathbf{1}_{len(\mathbf{x})}./\left(\mathbf{x}+\mathbf{a}\right)\right)^{\dot{\wedge}2}\right).$$

Equality constraint system with channel function (attracting towards the boundary)

$$\mathcal{D}_{eq}\left(\mathbf{x}\right) = \left(\log\left(|\mathbf{x}|+\mathbf{b}\right)\right)^{T}\mathbf{1}_{len(\mathbf{x})};$$

$$\boldsymbol{\nabla}_{\mathcal{D}_{eq}}\left[\mathbf{x}\right] = \mathbf{1}_{len(\mathbf{x})}./\left(\mathbf{x}+\mathbf{b}\dot{\times}\,\mathrm{sgn}\left(\mathbf{x}\right)\right);$$

$$\boldsymbol{\nabla}^{2}_{\mathcal{D}_{eq}}\left[\mathbf{x}\right] = -\mathrm{diag}\left(\left(\mathbf{1}_{len(\mathbf{x})}./\left(\mathbf{x}+\mathbf{b}\dot{\times}\,\mathrm{sgn}\left(\mathbf{x}\right)\right)\right)^{\dot{\wedge}2}\right),$$