Assimilation of Multiple Linearly Dependent Data Vectors

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Linearly dependent data vectors

Assume that we want to assimilate the data vectors $\{d_l\}_{l=1}^{L}$, where $\{d_l = B_l d_L\}_{l=1}^{L-1}$ and $\{B_l\}_{l=1}^{L-1}$ denotes a sequence of matrices

Linearly dependent data vectors

Main issue

Assume that we want to assimilate the data vectors $\{d_l\}_{l=1}^{L}$, where $\{d_l = B_l d_L\}_{l=1}^{L-1}$ and $\{B_l\}_{l=1}^{L-1}$ denotes a sequence of matrices

What is the appropriate way to assimilate such a data sequence, taking into account that some, but not necessarily all, information is used multiple times?

Outline

Motivation for considering linearly dependent data vectors

Relation to multiple data assimilation (MDA)

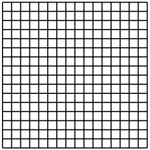
Brief recap of MDA condition (ensuring correct sampling in linear-Gaussian case)

Generalization of MDA condition to linearly dependent data vectors (PMDA condition)

PMDA condition in practice - some issues

Linearly dependent data vectors—example

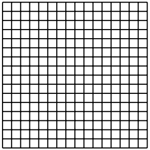
Data grid



Linearly dependent data vectors—example

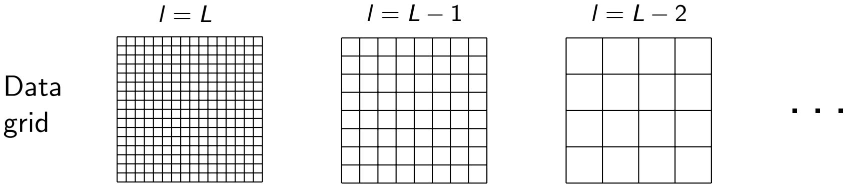
$I = L$

Data grid



Linearly dependent data vectors—example

Multilevel data

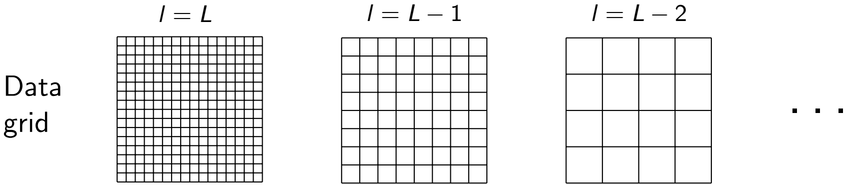


$$\{d_l = B_l d_L\}_{l=1}^{L-1}$$

With multilevel data, B_l denotes an averaging operator from level L to level l

Linearly dependent data vectors—example

Multilevel data



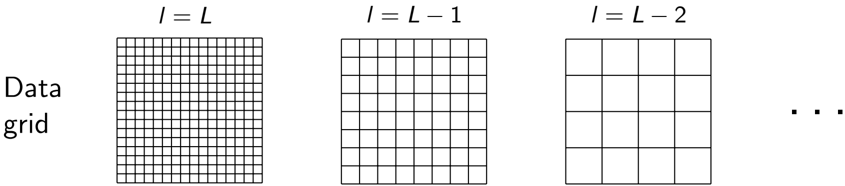
$$\{d_l = B_l d_L\}_{l=1}^{L-1}$$

With multilevel data, B_l denotes an averaging operator from level L to level l

Time-domain multilevel data is also a possibility

Multilevel data

Why bother?



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Multilevel data

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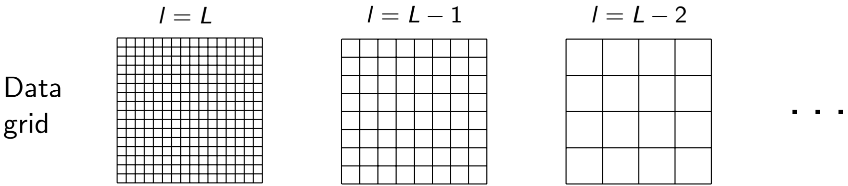

$$\{d_l = B_l d_L\}_{l=1}^{L-1}$$

Gradually introducing more and more information, like with sequential assimilation of $d_1, d_2, \ldots, d_L$, can be advantageous for nonlinear problems

Multilevel data

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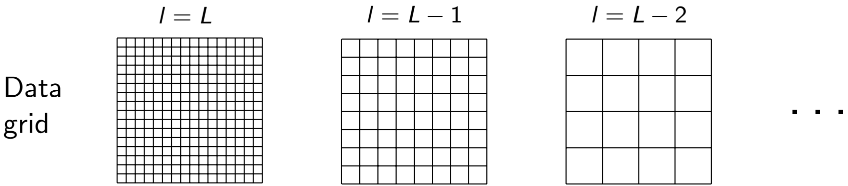


$$\{d_l = B_l d_L\}_{l=1}^{L-1}$$

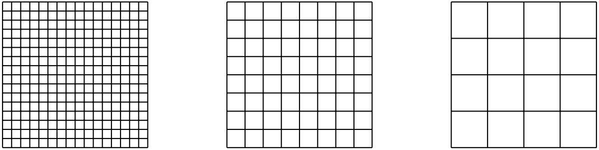
Gradually introducing more and more information, like with sequential assimilation of $d_1, d_2, \ldots, d_L$, can be advantageous for nonlinear problems

Multilevel data are required in order to correspond to results from multilevel simulations

Multilevel simulations

. . .

Sim. output grid



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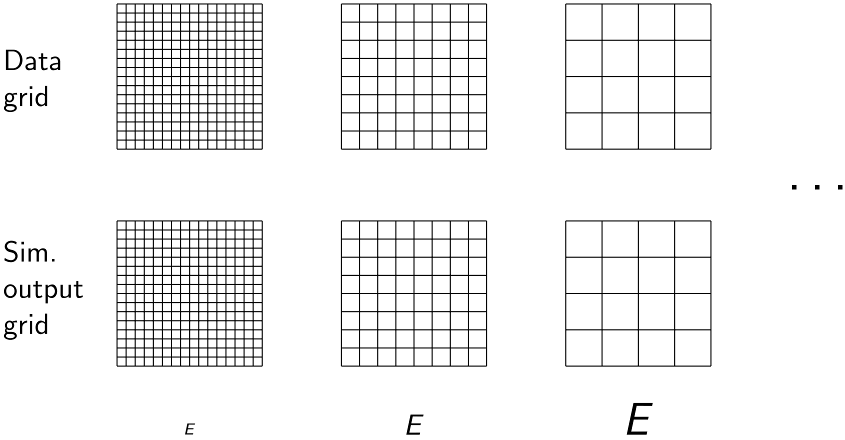
Multilevel simulations

...and corresponding multilevel data



Outline

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Brief recap of MDA condition (ensuring correct sampling in linear-Gaussian case)

Generalization of MDA condition to linearly dependent data vectors (PMDA condition)

PMDA condition in practice - some issues

Multiple data assimilation[1] (MDA)

Brief description

With MDA, the same data are assimilated multiple times. Since the data are reused, the data-error covariances must be inflated. The motivation for MDA is to improve performance on nonlinear problems by gradually introducing the available information in the data, leading to a sequence of smaller updates instead of a single large update

[1] Emerick and Reynolds, Computers & Geosci **55**, 2013

MDA

Multiple data
assimilation

$$\{d_l\}_{l=1}^{L}$$

$$\{d_l = d_L\}_{l=1}^{L-1}$$

Multiple use of
the same information

Abbreviation: MDA

MDA

... as a special case of assimilation of multiple linearly related data vectors

Multiple data assimilation	Assimilation of multiple linearly related data vectors
$\{d_l\}_{l=1}^{L}$	$\{d_l\}_{l=1}^{L}$
$\{d_l = d_L\}_{l=1}^{L-1}$	$\{d_l = B_l d_L\}_{l=1}^{L-1}$
Multiple use of the same information	Partially multiple use of the same information
Abbreviation: MDA	Abbreviation: PMDA (Partially MDA)

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MDA condition

Brief recap

While the motivation for MDA is to improve performance on nonlinear problems, it is desirable that it samples correctly from the posterior PDF for the parameter vector in the linear-Gaussian case

MDA condition

Brief recap

While the motivation for MDA is to improve performance on nonlinear problems, it is desirable that it samples correctly from the posterior PDF for the parameter vector in the linear-Gaussian case. This case can be analyzed using assembled quantities, where each row corresponds to an assimilation cycle

$$\delta = \begin{pmatrix} d_L \\ \vdots \\ d_L \end{pmatrix}$$

$$\Xi = \begin{pmatrix} C_L & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & C_L \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} G_L \\ \vdots \\ G_L \end{pmatrix}$$

MDA condition

Brief recap

While the motivation for MDA is to improve performance on nonlinear problems, it is desirable that it samples correctly from the posterior PDF for the parameter vector, m, in the linear-Gaussian case. This case can be analyzed using assembled quantities, where each row corresponds to an assimilation cycle. The analysis[2] leads to an inflated assembled covariance and the MDA condition for the inflation coefficients

$$\delta = \begin{pmatrix} d_L \\ \vdots \\ d_L \end{pmatrix} \qquad \Xi = \begin{pmatrix} \alpha_1 C_L & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \alpha_L C_L \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} G_L \\ \vdots \\ G_L \end{pmatrix} \qquad \sum_{l=1}^{L} \alpha_l^{-1} = 1$$

[2] Emerick and Reynolds, Computers & Geosci **55**, 2013

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MDA condition

Slight change of notation

To prepare for the description of the PMDA condition, which follows next, I use the subscript *MDA* for 'MDA quantities'

$$\delta_{MDA} = \begin{pmatrix} d_L \\ \vdots \\ d_L \end{pmatrix}$$

$$\Xi_{MDA} = \begin{pmatrix} \alpha_1 C_L & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \alpha_L C_L \end{pmatrix}$$

$$\Gamma_{MDA} = \begin{pmatrix} G_L \\ \vdots \\ G_L \end{pmatrix}$$

$$\sum_{l=1}^{L} \alpha_l^{-1} = 1$$

MDA condition

Slight change of notation

To prepare for the description of the PMDA condition, which follows next, I use the subscript *MDA* for 'MDA quantities', I introduce the coefficients $\{\lambda_l = \alpha_l^{1/2}\}_{l=1}^{L}$

$$\delta_{MDA} = \begin{pmatrix} d_L \\ \vdots \\ d_L \end{pmatrix}$$

$$\Xi_{MDA} = \begin{pmatrix} \lambda_1^2 C_L & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_L^2 C_L \end{pmatrix}$$

$$\Gamma_{MDA} = \begin{pmatrix} G_L \\ \vdots \\ G_L \end{pmatrix}$$

$$\sum_{l=1}^{L} \left(\lambda_l^2\right)^{-1} = 1$$

MDA condition

Slight change of notation

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$$\delta_{MDA} = \begin{pmatrix} d_L \\ \vdots \\ d_L \end{pmatrix} \qquad \Xi_{MDA} = \begin{pmatrix} \lambda_1^2 C_L & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_L^2 C_L \end{pmatrix}$$

$$\Gamma_{MDA} = \begin{pmatrix} G_L \\ \vdots \\ G_L \end{pmatrix} \qquad C_L^{-1} \sum_{l=1}^{L} \left(\lambda_l^2\right)^{-1} = C_L^{-1}$$

MDA condition

Slight change of notation

To prepare for the description of the PMDA condition, which follows next, I use the subscript *MDA* for 'MDA quantities', I introduce the coefficients $\{\lambda_l = \alpha_l^{1/2}\}_{l=1}^{L}$, I multiply the MDA condition by C_L^{-1}, and I reformulate the assembled data covariance and the MDA condition slightly

$$\delta_{MDA} = \begin{pmatrix} d_L \\ \vdots \\ d_L \end{pmatrix} \qquad \Xi_{MDA} = \begin{pmatrix} \lambda_1 C_L \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_L C_L \lambda_L \end{pmatrix}$$

$$\Gamma_{MDA} = \begin{pmatrix} G_L \\ \vdots \\ G_L \end{pmatrix} \qquad \sum_{l=1}^{L} (\lambda_l C_L \lambda_l)^{-1} = C_L^{-1}$$

MDA condition

$$\delta_{MDA} = \begin{pmatrix} d_L \\ \vdots \\ d_L \end{pmatrix}$$

$$\Xi_{MDA} = \begin{pmatrix} \lambda_1 C_L \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_L C_L \lambda_L \end{pmatrix}$$

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MDA condition

$$\delta_{MDA} = \begin{pmatrix} d_L \\ \vdots \\ d_L \end{pmatrix}$$

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$$\sum_{l=1}^{L} (\lambda_l C_L \lambda_l)^{-1} = C_L^{-1}$$

$$\delta_{PMDA} = \begin{pmatrix} d_1 \\ \vdots \\ d_L \end{pmatrix}$$

$$\Gamma_{PMDA} = \begin{pmatrix} G_1 \\ \vdots \\ G_L \end{pmatrix}$$

MDA condition and PMDA condition

$$\delta_{MDA} = \begin{pmatrix} d_L \\ \vdots \\ d_L \end{pmatrix}$$

$$\Xi_{MDA} = \begin{pmatrix} \lambda_1 C_L \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_L C_L \lambda_L \end{pmatrix}$$

$$\Gamma_{MDA} = \begin{pmatrix} G_L \\ \vdots \\ G_L \end{pmatrix}$$

$$\sum_{l=1}^{L} (\lambda_l C_L \lambda_l)^{-1} = C_L^{-1}$$

$$\delta_{PMDA} = \begin{pmatrix} d_1 \\ \vdots \\ d_L \end{pmatrix}$$

$$\Xi_{PMDA} = \begin{pmatrix} A_1 C_1 A_1^T & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_L C_L A_L^T \end{pmatrix}$$

$$\Gamma_{PMDA} = \begin{pmatrix} G_1 \\ \vdots \\ G_L \end{pmatrix}$$

$$\sum_{l=1}^{L} B_l^T \left(A_l C_l A_l^T\right)^{-1} B_l = C_L^{-1}$$

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PMDA condition in practice - some issues

PMDA condition in practice

Specification of Ξ_{PMDA}

$$\Xi_{PMDA} = \begin{pmatrix} A_1 C_1 A_1^T & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_L C_L A_L^T \end{pmatrix} \qquad \sum_{l=1}^{L} B_l^T \left(A_l C_l A_l^T\right)^{-1} B_l = C_L^{-1}$$

The specification of $\{\alpha_l\}_{l=1}^{L}$ in Ξ_{MDA} raises no other issue than how to make MDA perform optimally on a given nonlinear problem. Resolving this issue is not straightforward, but the specification of $\{A_l\}_{l=1}^{L}$ in Ξ_{PMDA} raises some issues in addition

PMDA condition in practice

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Before discussing these additional issues, note that since $\{d_l = B_l d_L\}_{l=1}^{L-1}$, it follows that $\{C_l = B_l C_L B_l^T\}_{l=1}^{L-1}$, leading to the following reformulated PMDA condition

$$\sum_{l=1}^{L-1} B_l^T \left(A_l B_l C_L B_l^T A_l^T\right)^{-1} B_l + \left(A_L C_L A_L^T\right)^{-1} = C_L^{-1}$$

PMDA condition in practice

Specification of Ξ_{PMDA}—some issues

$$\sum_{l=1}^{L-1} B_l^T \left(A_l B_l C_L B_l^T A_l^T\right)^{-1} B_l + \left(A_L C_L A_L^T\right)^{-1} = C_L^{-1}$$

PMDA condition in practice

Specification of Ξ_{PMDA}—some issues

$$\sum_{l=1}^{L-1} B_l^T \left(A_l B_l C_L B_l^T A_l^T\right)^{-1} B_l + \left(A_L C_L A_L^T\right)^{-1} = C_L^{-1}$$

All but one of the matrices $\{A_l\}_{l=1}^{L}$ can be specified freely, while the remaining one must be selected to fulfill the PMDA condition

PMDA condition in practice

Specification of Ξ_{PMDA}—some issues

$$\sum_{l=1}^{L-1} B_l^T \left(A_l B_l C_L B_l^T A_l^T\right)^{-1} B_l + \left(A_L C_L A_L^T\right)^{-1} = C_L^{-1}$$

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Solving the PMDA condition for one of the A_l's seems difficult

PMDA condition in practice

Specification of Ξ_{PMDA}—some issues

$$\sum_{l=1}^{L-1} B_l^T \left(A_l B_l C_L B_l^T A_l^T\right)^{-1} B_l + \left(A_L C_L A_L^T\right)^{-1} = C_L^{-1}$$

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Solving the PMDA condition for one of the A_l's seems difficult
Solving it for $A_L C_L A_L^T$ is, however, viable

$$A_L C_L A_L^T = \left(C_L^{-1} - \sum_{l=1}^{L-1} B_l^T \left(A_l B_l C_L B_l^T A_l^T\right)^{-1} B_l\right)^{-1}$$

PMDA condition in practice

Specification of Ξ_{PMDA}—some issues

$$\sum_{l=1}^{L-1} B_l^T \left(A_l B_l C_L B_l^T A_l^T\right)^{-1} B_l + \left(A_L C_L A_L^T\right)^{-1} = C_L^{-1}$$

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Solving the PMDA condition for one of the A_l's seems difficult
Solving it for $A_L C_L A_L^T$ is, however, viable

$$A_L C_L A_L^T = \left(I_L - C_L \sum_{l=1}^{L-1} B_l^T \left(A_l B_l C_L B_l^T A_l^T\right)^{-1} B_l\right)^{-1} C_L$$

PMDA condition in practice

Specification of Ξ_{PMDA}—a possibility

$$A_L C_L A_L^T = \left(I_L - C_L \sum_{l=1}^{L-1} B_l^T \left(A_l B_l C_L B_l^T A_l^T \right)^{-1} B_l \right)^{-1} C_L$$

PMDA condition in practice

Specification of Ξ_{PMDA}—a possibility

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Selecting $\{A_l = \alpha_l^{1/2} I_l\}_{l=1}^{L-1}$ leads to

$$A_L C_L A_L^T = \left(I_L - C_L \sum_{l=1}^{L-1} \alpha_l^{-1} B_l^T \left(B_l C_L B_l^T \right)^{-1} B_l \right)^{-1} C_L$$

PMDA condition in practice

Specification of Ξ_{PMDA}—a possibility

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PMDA condition in practice

Specification of Ξ_{PMDA}—a possibility

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One may then write

$$\Xi_{PMDA} = \begin{pmatrix} \Xi_{MDA}^{[1,L-1]} & 0 \\ 0 & (I_L - Q_L)^{-1} C_L \end{pmatrix}$$

PMDA condition in practice

Specification of Ξ_{PMDA}—a possibility with some issues

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PMDA condition in practice

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For a given matrix sequence $\{B_l\}_{l=1}^{L}$, one can risk selecting $\{\alpha_l\}_{l=1}^{L-1}$ such that $\left(I_L - Q_L\right)^{-1} C_L$ does not become a covariance matrix

PMDA condition in practice

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The matrix $I_L - Q_L$ can be computationally costly to invert for large problems

PMDA condition in practice

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The matrix $I_L - Q_L$ can be computationally costly to invert for large problems

Specifying sufficiently large elements in $\{\alpha_l\}_{l=1}^{L-1}$ will make $\|Q_L\|$ small enough that $(I_L - Q_L)^{-1} C_L$ becomes a covariance matrix, and it will allow for approximation of $(I_L - Q_L)^{-1}$ by a truncated Neumann series

PMDA condition in practice

Specification of Ξ_{PMDA}—a possibility with some issues

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The matrix $I_L - Q_L$ can be computationally costly to invert for large problems

Specifying sufficiently large elements in $\{\alpha_l\}_{l=1}^{L-1}$ will make $\|Q_L\|$ small enough that $(I_L - Q_L)^{-1} C_L$ becomes a covariance matrix, and it will allow for approximation of $(I_L - Q_L)^{-1}$ by a truncated Neumann series. Specifying too large elements in $\{\alpha_l\}_{l=1}^{L-1}$ will, however, effectively remove the influence of $\{d_l\}_{l=1}^{L-1}$ on the assimilation, which is unwanted. A balanced specification of $\{\alpha_l\}_{l=1}^{L-1}$ is therefore required

Summary

Assimilation of multiple linearly dependent data vectors incorporates use of some information multiple times (partially multiple data asssimilation (PMDA)). The corresponding data covariance matrices should therefore be modified.

A condition that the modified covariance matrices must satisfy in order to sample correctly in the linear-Gaussian case has been developed (Mannseth, in review). This PMDA condition is a generalization of the MDA condition (Emerick and Reynolds, Computers & Geosci 55, 2013) that the covariances must satisfy in the special case when a single data vector is assimilated multiple times

A simplified version of the PMDA condition has been proposed (Mannseth, in review). Also application of the simplified version involves both computational and accuracy issues

Acknowledgements

Partial financial support was provided by the research projects

4D Seismic History Matching (2015-2018), funded by Eni, Petrobras, the Research Council of Norway (RCN-Petromaks 2) and Total EP Norge

Assimilating 4D seismic data: big data into big models (2019-2022), funded by Aker BP, Equinor, Repsol, RCN-Petromaks 2 and Total EP Norge