An iterative ensemble Kalman filter in presence of additive model error

Marc Bocquet[1], Pavel Sakov[2], Jean-Matthieu Haussaire[1]

(1) CEREA, joint lab École des Ponts ParisTech and EdF R&D, Université Paris-Est, France
Institut Pierre-Simon Laplace
(2) Environment and Research Division, Bureau of Meteorology, Melbourne, Australia

(marc.bocquet@enpc.fr)

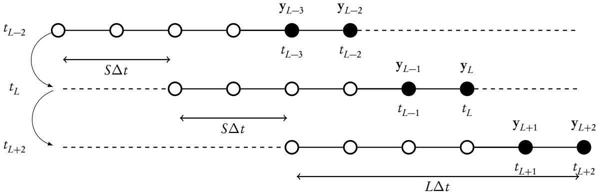
What this talk is about ...

▶ Iterative ensemble Kalman smoother (IEnKS): exemplar of nonlinear four-dimensional EnVar methods.

▶ It propagates the error statistics from one cycle to the next with the ensemble (errors of the day).

▶ It performs a 4D-Var analysis at each cycle (within the ensemble subspace).

▶ Typical cycling ($L = 6$, $S = 2$):



Variational analysis in ens. space → Posterior ens. generation → Ens. forecast

Cost functions

- General cost function over $[t_1, \ldots, t_L]$; weak-constraint formalism:

$$J_L(\mathbf{x}_1, \ldots, \mathbf{x}_L) = \|\mathbf{x}_1 - \mathbf{x}_1^{\mathrm{f}}\|^2_{(\mathbf{P}_1^{\mathrm{f}})^{-1}} + \sum_{i=1}^{L} \|\mathbf{y}_i - \mathscr{H}_i(\mathbf{x}_i)\|^2_{\mathbf{R}_i^{-1}} + \sum_{i=2}^{L} \|\mathbf{x}_i - \mathscr{M}_i(\mathbf{x}_{i-1})\|^2_{\mathbf{Q}_i^{-1}}.$$

- Configurations addressed in this talk:
  - The case $L = 1$, $S = 1$, the so-called iterative ensemble Kalman filter → IEnKF:

$$J_L(\mathbf{x}_1, \mathbf{x}_2) = \|\mathbf{x}_1 - \mathbf{x}_1^{\mathrm{f}}\|^2_{(\mathbf{P}_1^{\mathrm{f}})^{-1}} + \|\mathbf{y}_2 - \mathscr{H}_2 \circ \mathscr{M}_2(\mathbf{x}_1)\|^2_{\mathbf{R}_2^{-1}}.$$

  - The IEnKF but, now, with additive model error → IEnKF-Q :

$$J_L(\mathbf{x}_1, \mathbf{x}_2) = \|\mathbf{x}_1 - \mathbf{x}_1^{\mathrm{f}}\|^2_{(\mathbf{P}_1^{\mathrm{f}})^{-1}} + \|\mathbf{y}_2 - \mathscr{H}_2(\mathbf{x}_2)\|^2_{\mathbf{R}_2^{-1}} + \|\mathbf{x}_2 - \mathscr{M}_2(\mathbf{x}_1)\|^2_{\mathbf{Q}_2^{-1}}.$$

  - The *linearized* case $L + 1 = S$ with additive model error, called the asynchronous ensemble Kalman filter → AEnKF.

P. Sakov, J.-M. Haussaire, and M. Bocquet, *An iterative ensemble Kalman filter in presence of additive model error*, Q. J. R. Meteorol. Soc., 0 (2017), pp. 0–0. Submitted

Outline

Iterative ensemble Kalman filter: a Bayesian standpoint

- Gaussian assumption for the prior:

$$p(\mathbf{x}_1) = n\left(\mathbf{x}_1|\overline{\mathbf{x}}_1, \mathbf{P}_1\right).$$

- Forecast under perfect model assumption:

$$p(\mathbf{x}_2|\mathbf{x}_1) \propto \delta\left\{\mathbf{x}_2 - \mathscr{M}_2(\mathbf{x}_1)\right\}.$$

- Likelihood used in the analysis:

$$p(\mathbf{y}_2|\mathbf{x}_2) = n\left(\mathbf{y}_2 - H_2(\mathbf{x}_2)|\mathbf{0}, \mathbf{R}_2\right).$$

- (Full cycle) analysis of the initial condition $\mathbf{x}_1$:

$$\begin{aligned} p(\mathbf{x}_1|\mathbf{y}_2) &\propto p(\mathbf{y}_2|\mathbf{x}_1)p(\mathbf{x}_1) \\ &\propto p(\mathbf{y}_2|\mathbf{x}_2 = \mathscr{M}_2(\mathbf{x}_1))p(\mathbf{x}_1). \end{aligned}$$

- Analysis (forecast!) of the filtering distribution:

$$\begin{aligned} p(\mathbf{x}_2|\mathbf{y}_2) &= \int \mathrm{d}\mathbf{x}_1\, p(\mathbf{x}_2|\mathbf{x}_1, \mathbf{y}_2)p(\mathbf{x}_1|\mathbf{y}_2) \\ &= \int \mathrm{d}\mathbf{x}_1\, \delta\left\{\mathbf{x}_2 - \mathscr{M}_2(\mathbf{x}_1)\right\} p(\mathbf{x}_1|\mathbf{y}_2). \end{aligned}$$

M. Bocquet and P. Sakov, *An iterative ensemble Kalman smoother*, Q. J. R. Meteorol. Soc., 140 (2014), pp. 1521–1535

Iterative ensemble Kalman filter: a variational standpoint

- Analysis IEnKF cost function in state space $p(\mathbf{x}_1|\mathbf{y}_2) \propto \exp\left(-\mathscr{J}(\mathbf{x}_1)\right)$:

$$\mathscr{J}(\mathbf{x}_1) = \frac{1}{2}\|\mathbf{y}_2 - \mathscr{H}_2 \circ \mathscr{M}_2(\mathbf{x}_1)\|^2_{\mathbf{R}_2^{-1}} + \frac{1}{2}\|\mathbf{x}_1 - \overline{\mathbf{x}}_1\|^2_{\mathbf{P}_1^{-1}}.$$

- Reduced scheme in ensemble subspace, $\mathbf{x}_1 = \overline{\mathbf{x}}_1 + \mathbf{A}_1\mathbf{w}_1$, where $\mathbf{A}_1$ is the normalized ensemble anomaly matrix:

$$\widetilde{\mathscr{J}}(\mathbf{w}_1) = \mathscr{J}(\overline{\mathbf{x}}_1 + \mathbf{A}_1\mathbf{w}_1).$$

- IEnKF cost function in ensemble space:

$$\widetilde{\mathscr{J}}(\mathbf{w}_1) = \frac{1}{2}\|\mathbf{y}_2 - \mathscr{H}_2 \circ \mathscr{M}_2\left(\overline{\mathbf{x}}_1 + \mathbf{A}_1\mathbf{w}_1\right)\|^2_{\mathbf{R}_2^{-1}} + \frac{1}{2}\|\mathbf{w}_1\|^2.$$

P. Sakov, D. S. Oliver, and L. Bertino, *An iterative EnKF for strongly nonlinear systems*, Mon. Wea. Rev., 140 (2012), pp. 1988–2004

M. Bocquet and P. Sakov, *Combining inflation-free and iterative ensemble Kalman filters for strongly nonlinear systems*, Nonlin. Processes Geophys., 19 (2012), pp. 383–399

Iterative ensemble Kalman filter: minimization scheme

- As a variational reduced method, one can use Gauss-Newton [Sakov et al., 2012], Levenberg-Marquardt [Bocquet and Sakov, 2012; Chen and Oliver, 2012], etc, minimization schemes (not limited to quasi-Newton).

- Gauss-Newton scheme:

$$
\begin{aligned}
\mathbf{w}_1^{(j+1)} &= \mathbf{w}_1^{(j)} - \widetilde{\mathscr{H}}_{(j)}^{-1} \nabla \widetilde{\mathscr{J}}_{(j)}(\mathbf{w}_1^{(j)}), \\
\mathbf{x}_1^{(j)} &= \overline{\mathbf{x}}_1 + \mathbf{A}_1 \mathbf{w}_1^{(j)}, \\
\nabla \widetilde{\mathscr{J}}_{(j)} &= \mathbf{w}_1^{(j)} - \mathbf{Y}_{(j)}^{\mathrm{T}} \mathbf{R}_2^{-1} \left(\mathbf{y}_2 - \mathscr{H}_2 \circ \mathscr{M}_2(\mathbf{x}_1^{(j)}) \right), \\
\widetilde{\mathscr{H}}_{(j)} &= \mathbf{I}_N + \mathbf{Y}_{(j)}^{\mathrm{T}} \mathbf{R}_2^{-1} \mathbf{Y}_{(j)}, \\
\mathbf{Y}_{(j)} &= [\mathscr{H}_2 \circ \mathscr{M}_2]'_{|\mathbf{x}_1^{(j)}} \mathbf{A}_1 .
\end{aligned}
$$

Iterative ensemble Kalman filter: computing the sensitivities

- Sensitivities $\mathbf{Y}_{(p)}$ computed by ensemble propagation without TLM and adjoint ([Gu and Oliver, 2007; Liu et al., 2008; Buehner et al., 2010])

- First alternative [Sakov et al., 2012]: the transform scheme. The ensemble is preconditioned before its propagation using the ensemble transform

$$\mathbf{T}_{(j)} = \left(\mathbf{I}_N + \mathbf{Y}_{(j)}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{Y}_{(j)}\right)^{-1/2},$$

obtained at the previous iteration. The inverse transformation is applied after propagation.

- Second alternative [Bocquet and Sakov, 2012]: the bundle scheme. It simply mimics the action of the tangent linear by finite difference:

$$\mathbf{Y}_{(j)} \approx \frac{1}{\varepsilon}\mathscr{H}_2 \circ \mathscr{M}_2\left(\mathbf{x}^{(j)}\mathbf{1}^{\mathrm{T}} + \varepsilon\mathbf{A}_1\right)\left(\mathbf{I}_N - \frac{\mathbf{1}\mathbf{1}^{\mathrm{T}}}{N}\right).$$

Outline

IEnKF-Q: formulation

- Analysis cost function:

$$\mathcal{J}(\mathbf{x}_1,\mathbf{x}_2) = \left\|\mathbf{x}_1-\mathbf{x}_1^{\mathrm{a}}\right\|^2_{(\mathbf{P}_1^{\mathrm{a}})^{-1}} + \left\|\mathbf{y}_2-\mathscr{H}(\mathbf{x}_2)\right\|^2_{\mathbf{R}^{-1}} + \left\|\mathbf{x}_2-\mathscr{M}(\mathbf{x}_1)\right\|^2_{\mathbf{Q}^{-1}}.$$

- Ensemble subspace representation:

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{x}_1^{\mathrm{a}} + \mathbf{A}_1^{\mathrm{a}}\mathbf{u}, \qquad & \mathbf{A}_1^{\mathrm{a}}(\mathbf{A}_1^{\mathrm{a}})^{\mathrm{T}} &= \mathbf{P}_1^{\mathrm{a}}, \qquad & \mathbf{A}_1^{\mathrm{a}}\mathbf{1} &= \mathbf{0}, \\ \mathbf{x}_2 &= \mathscr{M}(\mathbf{x}_1) + \mathbf{A}_2^{q}\mathbf{v}, \qquad & \mathbf{A}_2^{q}(\mathbf{A}_2^{q})^{\mathrm{T}} &= \mathbf{Q}, \qquad & \mathbf{A}_2^{q}\mathbf{1} &= \mathbf{0}. \end{aligned}$$

- Cost function in ensemble subspace:

$$\mathcal{J}(\mathbf{u},\mathbf{v}) = \mathbf{u}^{\mathrm{T}}\mathbf{u} + \mathbf{v}^{\mathrm{T}}\mathbf{v} + \left\|\mathbf{y}_2-\mathscr{H}(\mathbf{x}_2)\right\|^2_{\mathbf{R}^{-1}}.$$

IEnKF-Q: formulation

- Compactification:

$$\mathbf{w} \equiv \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \quad \Longrightarrow \quad J(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{w} + \|\mathbf{y}_2 - \mathscr{H}(\mathbf{x}_2)\|^2_{\mathbf{R}^{-1}} .$$

- Condition of zero gradient:

$$\mathbf{w} - (\mathbf{HA})^{\mathrm{T}}\mathbf{R}^{-1}\left[\mathbf{y}_2 - \mathscr{H}(\mathbf{x}_2)\right] = \mathbf{0},$$

where

$$\mathbf{A} \equiv [\mathbf{M}\mathbf{A}_1^{\mathrm{a}}, \mathbf{A}_2^{q}], \quad \mathbf{H} \equiv \nabla\mathscr{H}(\mathbf{x}_2), \quad \mathbf{M} \equiv \nabla\mathscr{M}(\mathbf{x}_1).$$

- The cost function can be minimized using a Gauss-Newton method

$$\mathbf{w}^{i+1} = \mathbf{w}^{i} - \mathbf{D}^{i}\nabla J(\mathbf{w}^{i}),$$

where the inverse Hessian is approximated as

$$\mathbf{D}^{i} \approx \left[\mathbf{I} + (\mathbf{H}^{i}\mathbf{A}^{i})^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}^{i}\mathbf{A}^{i}\right]^{-1}.$$

IEnKF-Q: formulation

- Posterior anomalies:

$$\delta \mathbf{x}_1 = \mathbf{A}_1^{\mathrm{a}}\,\delta\mathbf{u}, \qquad \delta\mathbf{x}_2 = \mathbf{M}\mathbf{A}_1^{\mathrm{a}}\,\delta\mathbf{u} + \mathbf{A}_2^{q}\,\delta\mathbf{v},$$

- Updated perturbations over $[t_1, t_2]$:

$$\mathbf{A}_2^{\mathrm{a}}(\mathbf{A}_2^{\mathrm{a}})^{\mathrm{T}} = \mathrm{E}[\delta\mathbf{x}_2^{\star}(\delta\mathbf{x}_2^{\star})^{\mathrm{T}}] = \mathbf{A}^{\star}\mathrm{E}[\mathbf{w}^{\star}(\mathbf{w}^{\star})^{\mathrm{T}}](\mathbf{A}^{\star})^{\mathrm{T}} = \mathbf{A}^{\star}\mathbf{D}^{\star}(\mathbf{A}^{\star})^{\mathrm{T}},$$

which implies

$$\mathbf{A}_2^{\mathrm{a}} = \mathbf{A}^{\star}(\mathbf{D}^{\star})^{1/2} = \mathbf{A}^{\star}\left[\mathbf{I} + (\mathbf{H}^{\star}\mathbf{A}^{\star})^{\mathrm{T}}(\mathbf{R})^{-1}\mathbf{H}^{\star}\mathbf{A}^{\star}\right]^{-1/2}.$$

- Updated (smoothed) perturbations at t_1:

$$\mathbf{A}_1^{\mathrm{s}}(\mathbf{A}_1^{\mathrm{s}})^{\mathrm{T}} = \mathrm{E}[\delta\mathbf{x}_1^{\star}(\delta\mathbf{x}_1^{\star})^{\mathrm{T}}] = \mathbf{A}_1^{\mathrm{a}}\mathrm{E}[\mathbf{u}^{\star}(\mathbf{u}^{\star})^{\mathrm{T}}](\mathbf{A}_1^{\mathrm{a}})^{\mathrm{T}},$$

which implies

$$\mathbf{A}_1^{\mathrm{s}} = \mathbf{A}_1^{\mathrm{a}}\,(\mathbf{D}^{\star}_{1:m,1:m})^{1/2}.$$

IEnKF-Q: Decoupling

- In all generality:

$$J(\mathbf{x}_1,\mathbf{x}_2) = -2\ln p(\mathbf{x}_1,\mathbf{x}_2|\mathbf{y}_2) = -2\ln p(\mathbf{x}_2|\mathbf{x}_1,\mathbf{y}_2)p(\mathbf{x}_1|\mathbf{y}_2).$$

- If the observation operator $\mathscr{H}$ is linear:

$$-2\ln p(\mathbf{x}_1|\mathbf{y}_2) = \|\mathbf{x}_1 - \mathbf{x}_1^{\mathrm{a}}\|^2_{(\mathbf{P}_1^{\mathrm{a}})^{-1}} + \|\mathbf{y}_2 - \mathscr{H}\circ\mathscr{M}(\mathbf{x}_1)\|^2_{(\mathbf{R}+\mathbf{HQH}^{\mathrm{T}})^{-1}} + c_1,$$

and

$$-2\ln p(\mathbf{x}_2|\mathbf{x}_1,\mathbf{y}_2) = \|\mathbf{x}_2 - \mathscr{M}(\mathbf{x}_1) - \mathbf{QH}^{\mathrm{T}}(\mathbf{R}+\mathbf{HQH}^{\mathrm{T}})^{-1}\left[\mathbf{y}_2 - \mathscr{H}\circ\mathscr{M}(\mathbf{x}_1)\right]\|^2_{\mathbf{Q}^{-1}+\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}} + c_2,$$

- Then the MAP of $J(\mathbf{x}_1,\mathbf{x}_2)$ can be computed in two steps:
  - Minimize $-2\ln p(\mathbf{x}_1|\mathbf{y}_2)$ over $\mathbf{x}_1$ just like the IEnKF in the absence of model error but with $\mathbf{R}\to\mathbf{R}+\mathbf{HQH}^{\mathrm{T}}$.
  - The MAP of $-2\ln p(\mathbf{x}_2|\mathbf{x}_1^\star,\mathbf{y}_2)$ is then directly given by:

$$\mathbf{x}_2^\star = \mathscr{M}(\mathbf{x}_1^\star) + \mathbf{QH}^{\mathrm{T}}\left(\mathbf{R}+\mathbf{HQH}^{\mathrm{T}}\right)^{-1}\left[\mathbf{y}_2 - \mathscr{H}\circ\mathscr{M}(\mathbf{x}_1^\star)\right].$$

IEnKF-Q: Decoupling

- This decoupling also implies the decoupling of $(\mathbf{u}, \mathbf{v})$:

$$\mathbf{u}^{i+1} - \mathbf{u}^{i} = \mathbf{D}_u^i \left\{ (\mathbf{H}\mathbf{M}^i\mathbf{A}_1^{\mathrm{a}})^{\mathrm{T}} (\mathbf{R}_u^i)^{-1} \times \left[\mathbf{y}_2 - \mathscr{H} \circ \mathscr{M}(\mathbf{x}_1^{\mathrm{a}} + \mathbf{A}_1^{\mathrm{a}}\mathbf{u}^i) \right] - \mathbf{u}^i \right\}.$$

$$\mathbf{v}^\star = \mathbf{D}_v^\star (\mathbf{H}\mathbf{A}_2^q)^{\mathrm{T}} (\mathbf{R}_v^\star)^{-1} \left[\mathbf{y}_2 - \mathscr{H}(\mathbf{x}_2^\star) + \mathbf{H}\mathbf{M}^\star\mathbf{A}_1^{\mathrm{a}}\mathbf{u}^\star \right].$$

- However, this decoupling does not convey to the perturbations update!

- The same decoupling is used in particle filtering [Doucet et al., 2000] to build the optimal importance proposal particle filter.

IEnKF-Q: algorithm

$$\textbf{function } [\mathbf{E}_2] = \textbf{ienkf\_cycle}(\mathbf{E}_1^{\mathrm{a}}, \mathbf{A}_2^{q}, \mathbf{y}_2, \mathbf{R}, \mathscr{M}, \mathscr{H})$$

$$\quad \mathbf{x}_1^{\mathrm{a}} = \mathbf{E}_1^{\mathrm{a}} \mathbf{1}/m$$

$$\quad \mathbf{A}_1^{\mathrm{a}} = (\mathbf{E}_1^{\mathrm{a}} - \mathbf{x}_1^{\mathrm{a}} \mathbf{1}^{\mathrm{T}})/\sqrt{m-1}$$

$$\quad \mathbf{D} = \mathbf{I}, \quad \mathbf{w} = \mathbf{0}$$

$$\quad \textbf{repeat}$$

$$\qquad \mathbf{x}_1 = \mathbf{x}_1^{\mathrm{a}} + \mathbf{A}_1^{\mathrm{a}} \mathbf{w}_{1:m}$$

$$\qquad \mathbf{T} = (\mathbf{D}_{1:m,1:m})^{1/2}$$

$$\qquad \mathbf{E}_1 = \mathbf{x}_1 \mathbf{1}^{\mathrm{T}} + \mathbf{A}_1^{\mathrm{a}} \mathbf{T} \sqrt{m-1}$$

$$\qquad \mathbf{E}_2 = \mathscr{M}(\mathbf{E}_1)$$

$$\qquad \mathbf{HA}_2 = \mathscr{H}(\mathbf{E}_2)(\mathbf{I} - \mathbf{1}\mathbf{1}^{\mathrm{T}}/m)\mathbf{T}^{-1}/\sqrt{m-1}$$

$$\qquad \mathbf{HA}_2^{q} = \mathscr{H}(\mathbf{E}_2 \mathbf{1}\mathbf{1}^{\mathrm{T}}/m + \mathbf{A}_2^{q}\sqrt{m_q - 1})(\mathbf{I} - \mathbf{1}\mathbf{1}^{\mathrm{T}}/m_q)/\sqrt{m_q - 1}$$

$$\qquad \mathbf{HA} = [\mathbf{HA}_2, \mathbf{HA}_2^{q}]$$

$$\qquad \mathbf{x}_2 = \mathbf{E}_2 \mathbf{1}/m + \mathbf{A}_2^{q} \mathbf{w}_{m+1:m+m_q}$$

$$\qquad \nabla J = \mathbf{w} - (\mathbf{HA})^{\mathrm{T}} \mathbf{R}^{-1} [\mathbf{y}_2 - \mathscr{H}(\mathbf{x}_2)]$$

$$\qquad \mathbf{D} = [\mathbf{I} + (\mathbf{HA})^{\mathrm{T}} \mathbf{R}^{-1} \mathbf{HA}]^{-1}$$

$$\qquad \Delta \mathbf{w} = -\mathbf{D} \nabla J$$

$$\qquad \mathbf{w} = \mathbf{w} + \Delta \mathbf{w}$$

$$\quad \textbf{until } |\Delta \mathbf{w}| < \varepsilon$$

$$\quad \mathbf{A}_2 = \mathbf{E}_2 (\mathbf{I} - \mathbf{1}\mathbf{1}^{\mathrm{T}}/m)\, \mathbf{T}^{-1}$$

$$\quad \mathbf{A} = [\mathbf{A}_2/\sqrt{m-1}, \mathbf{A}_2^{q}]\, \mathbf{D}^{1/2}$$

$$\quad \mathbf{A}_2 = \mathrm{SR}(\mathbf{A}, m)\sqrt{m-1}$$

$$\quad \mathbf{E}_2 = \mathbf{x}_2 \mathbf{1}^{\mathrm{T}} + (1+\delta)\mathbf{A}_2$$

$$\textbf{end function}$$

Outline

IEnKF-Q: numerical experiments

- Experiments performed on the Lorenz-95 model. Fully observed: $\mathbf{H} = \mathbf{I}$, $\mathbf{R} = \mathbf{I}$. We choose $m_q = 41$, so that $\mathbf{Q}$ is full rank.

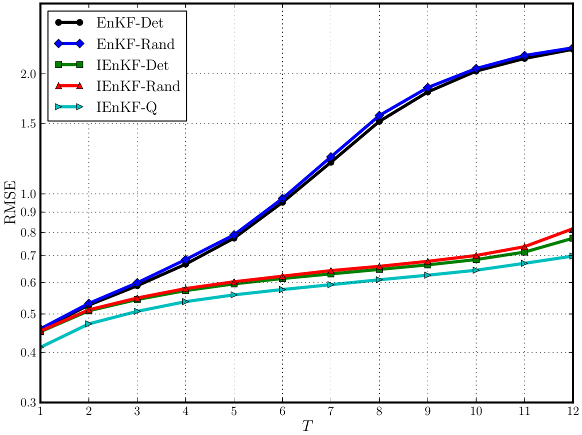
- Random mean-preserving rotations of the ensemble anomalies are sometimes applied to the IEnKF-Q, typically in the very weak model error regime.

- Comparisons with EnKF + accounting for $\mathbf{Q}$ and IEnKF + accounting for $\mathbf{Q}$:
  - [Rand] Stochastic approach: $\mathbf{A}_2^{\mathrm{f}} = \mathbf{M}\mathbf{A}_1^{\mathrm{a}} + \mathbf{Q}^{1/2}\boldsymbol{\Xi}$,
  - [Det] Deterministic approach: $\mathbf{A}_2^{\mathrm{f}} = \mathbf{A}\left[\mathbf{I} + \mathbf{A}^{\dagger}\mathbf{Q}(\mathbf{A}^{\dagger})^{\mathrm{T}}\right]^{1/2}$, with $\mathbf{A} = \mathbf{M}\mathbf{A}_1^{\mathrm{a}}$.

E. N. Lorenz and K. A. Emanuel, *Optimal sites for supplementary weather observations: simulation with a small model*, J. Atmos. Sci., 55 (1998), pp. 399–414

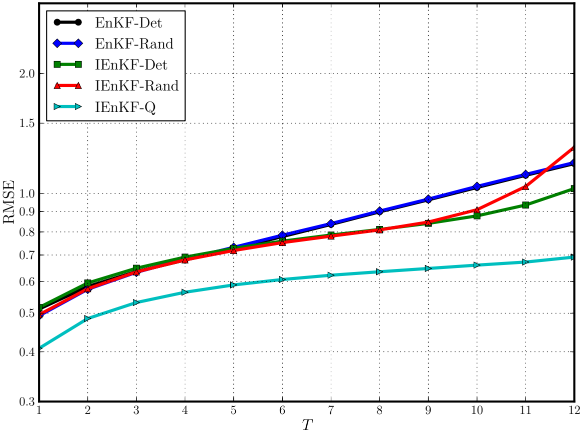
P. N. Raanes, A. Carrassi, and L. Bertino, *Extending the square root method to account for additive forecast noise in ensemble methods*, Mon. Wea. Rev., 143 (2015), pp. 3857–38730

Test 1: nonlinearity



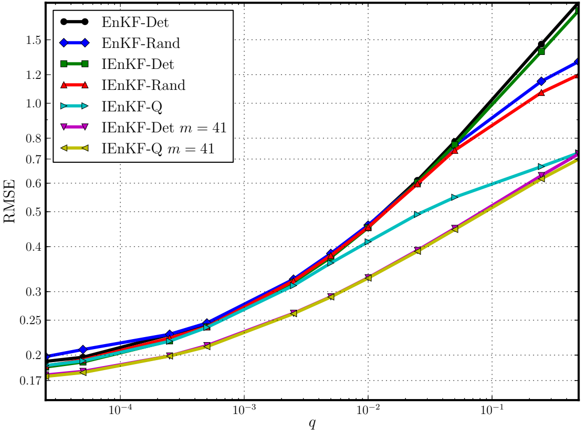
- $\mathbf{Q} = 0.01\,T\,\mathbf{I}$, $m = 20$.

Test 1: nonlinearity (non-diagonal $\mathbf{Q}$)



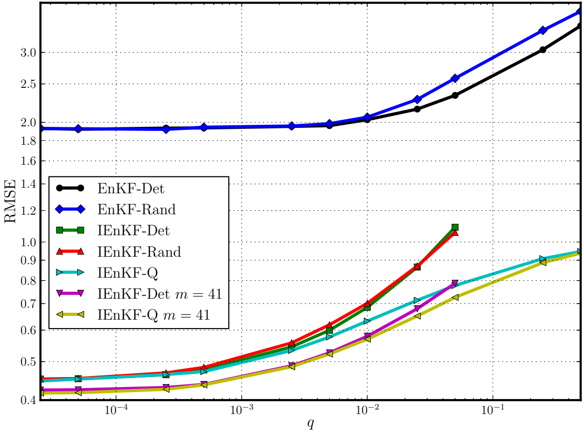
- $[\mathbf{Q}]_{ij} = 0.05\,T(\exp[-d^2(i,j)/30]) + 0.1\delta_{ij},\ m = 30$

Test 2: model noise magnitude



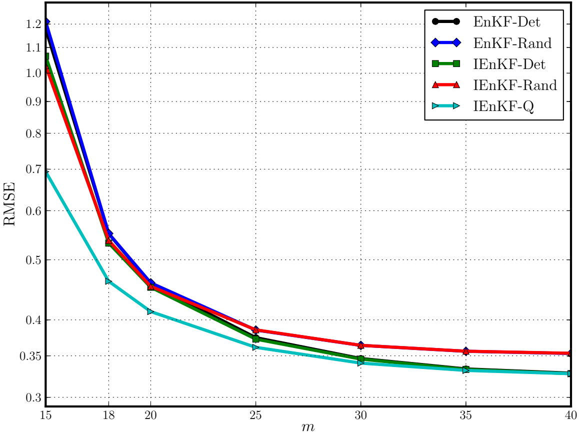
- $Q = qT\mathbf{I},\ T = 1,\ m = 20.$

Test 2: model noise magnitude



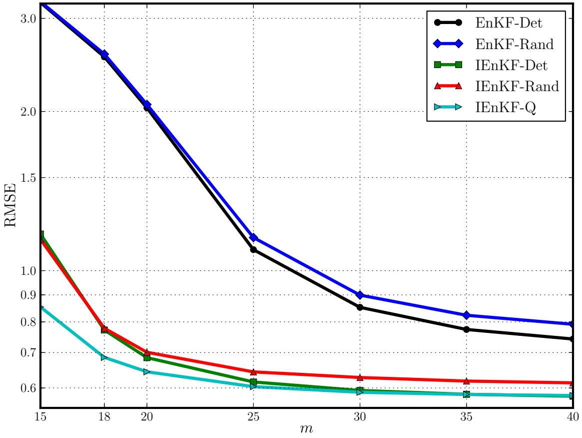
- $Q = qT\mathbf{I}$, $T = 10$, $m = 20$.

Test 3: ensemble size



- $Q = 0.01\,T\mathbf{I}$, $T = 1$.

Test 3: ensemble size



- $Q = 0.01\,T\mathbf{I}$, $T = 10$.

Outline

Asynchronous data assimilation for the EnKF

▶ How to simply and efficiently assimilate observations in between two update steps of the EnKF (linear order)?

B. R. Hunt, E. J. Kostelich, and I. Szunyogh, *Efficient data assimilation for spatiotemporal chaos: A local ensemble transform Kalman filter*, Physica D, 230 (2007), pp. 112–126

P. Sakov, G. Evensen, and L. Bertino, *Asynchronous data assimilation with the EnKF*, Tellus A, 62 (2010), pp. 24–29

▶ How to do so in presence of additive model error (linear order, $L \to k$)?

$$J(\mathbf{x}_0, \ldots, \mathbf{x}_k) = \|\mathbf{x}_0 - \mathbf{x}_0^a\|^2_{(\mathbf{P}_0^a)^{-1}} + \sum_{i=1}^{k} \|\mathbf{y}_i - \mathscr{H}_i(\mathbf{x}_i)\|^2_{\mathbf{R}_i^{-1}} + \sum_{i=1}^{k} \|\mathbf{x}_i - \mathscr{M}_{i-1\to i}(\mathbf{x}_{i-1})\|^2_{\mathbf{Q}_i^{-1}} .$$

P. Sakov and M. Bocquet, *Asynchronous data assimilation with the EnKF in presence of additive model error*, Tellus A, 0 (2017), pp. 0–0. in preparation

Asynchronous data assimilation for the EnKF

- Ensemble subspace representation, for $i = 1, \ldots, k$:

$$\mathbf{x}_0 = \mathbf{x}_0^a + \mathbf{A}_0^a \mathbf{w}_0, \quad \mathbf{A}_0^a (\mathbf{A}_0^a)^{\mathrm{T}} = \mathbf{P}_0^a, \quad \mathbf{A}_0^a \mathbf{1} = 0$$
$$\mathbf{x}_i = \mathscr{M}_{i-1 \to i}(\mathbf{x}_{i-1}) + \mathbf{A}_i^q \mathbf{w}_i, \quad \mathbf{A}_i^q (\mathbf{A}_i^q)^{\mathrm{T}} = \mathbf{Q}_i, \quad \mathbf{A}_i^q \mathbf{1} = 0$$

- Compactification:

$$\mathbf{w} \equiv \mathrm{vec}\left(\mathbf{w}_0, \ldots, \mathbf{w}_k\right).$$

- Cost function in ensemble subspace:

$$\tilde{J}(\mathbf{w}) = \mathbf{w}^{\mathrm{T}} \mathbf{w} + \| \mathbf{y} - \mathscr{H}(\mathbf{x}) \|^2_{\mathbf{R}^{-1}}.$$

Asynchronous data assimilation for the EnKF

- Linearization (Gauss-Newton implied):

$$\mathbf{x} = \mathbf{x}^f + \mathbf{A}\mathbf{w} + O(\|\mathbf{w}\|^2),$$

with

$$\mathbf{x}^f \equiv \mathrm{vec}\left(\{\mathscr{M}_{0\to i}(\mathbf{x}_0^a)\}_{i=0,\ldots,k}\right),$$

and

$$\mathbf{A} \equiv \mathrm{vec}(\mathbf{A}_0, \ldots, \mathbf{A}_k),$$

$$\mathbf{A}_i \equiv \begin{cases} [\mathbf{A}_0^a, \mathbf{0}], & i = 0 \\ [\mathbf{M}_{i-1\to i}\mathbf{A}_{i-1}, \mathbf{A}_i^q, \mathbf{0}], & i = 1, \ldots, k-1, \\ [\mathbf{M}_{k-1\to k}\mathbf{A}_{k-1}, \mathbf{A}_k^q], & i = k. \end{cases}$$

Asynchronous data assimilation for the EnKF

- Cost function expansion:

$$\tilde{J}(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{w} + \left\| \mathbf{y} - \mathscr{H}(\mathbf{x}^f) - \mathbf{Y}\mathbf{w} + O(\|\mathbf{w}\|^2) \right\|^2_{\mathbf{R}^{-1}},$$

where $\mathbf{Y} \equiv \mathrm{vec}\left(\{\mathbf{H}_i\mathbf{A}_i\}_{i=1,\ldots,k}\right)$.

- Linear order analysis (AEnKF):

$$\begin{aligned}
\mathbf{x}^\star &= \mathbf{x}^f + \mathbf{A}\mathbf{w}^\star, \\
\mathbf{A}^\star &= \mathbf{A}\mathbf{T}, \quad \mathbf{T} = \mathbf{D}^{-1/2}\mathbf{U}, \\
\mathbf{w}^\star &= \mathbf{D}^{-1}\mathbf{Y}^{\mathrm{T}}\mathbf{R}^{-1}\left[\mathbf{y} - \mathscr{H}(\mathbf{x}^f)\right], \\
\mathbf{D} &\equiv \mathbf{I} + \mathbf{Y}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{Y}.
\end{aligned}$$

- The computation of $\mathbf{A}_i$ and $\mathbf{Y}$ can also be extrapolated to mild nonlinearity.

P. Sakov and M. Bocquet, *Asynchronous data assimilation with the EnKF in presence of additive model error*, Tellus A, 0 (2017), pp. 0–0. in preparation

Outline

Conclusions

- We have extended the iterative ensemble Kalman filter (IEnKF) to iterative ensemble Kalman filter in presence of model error (IEnKF-Q).
- It consistently outperforms ad hoc schemes that incorporate model error into the IEnKF with the L95 model, and any other EnKF-based scheme.
- We have extended the asynchronous ensemble Kalman filter (AEnKF) to the asynchronous ensemble Kalman filter in presence of model error (AEnKF-Q).
- In practice, one would have to estimate $\mathbf{Q}$ on top of these developments. A currently flourishing topic!

Final word

Thank you for your attention!

References

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