Geostatistical spatial-temporal covariance modelling

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Space-time random function

Let $Z(\mathbf{x}, t)$ with $(\mathbf{x}, t) \in \mathbb{R}^d \times \mathbb{R}$ be a space-time random function.

- Physically we have a clear-cut separation between the spatial and time dimensions.

Assumptions about space-time covariance functions

Common simplifying assumptions about the space-time covariance:

Separability:

$$\mathrm{cov}(Z(\mathbf{x}_1, t_1), Z(\mathbf{x}_2, t_2)) = C_S(\mathbf{x}_1, \mathbf{x}_2) \cdot C_T(t_1, t_2)$$

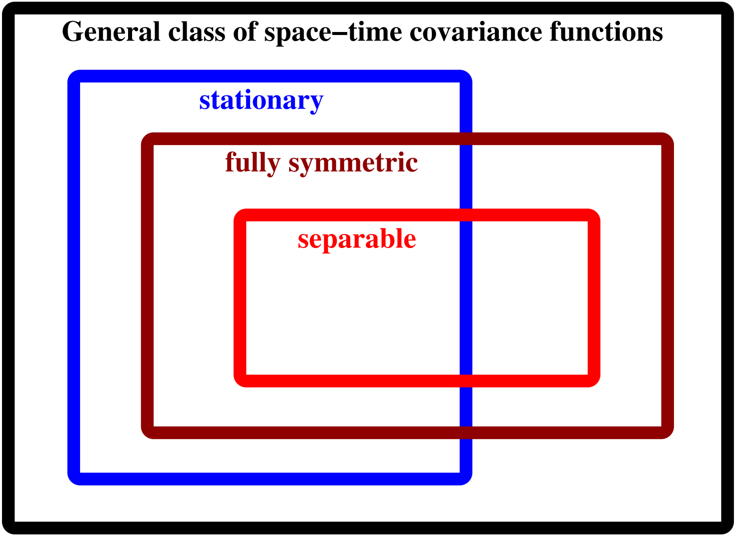
Full symmetry:

$$\mathrm{cov}(Z(\mathbf{x}_1, t_1), Z(\mathbf{x}_2, t_2)) = \mathrm{cov}(Z(\mathbf{x}_1, t_2), Z(\mathbf{x}_2, t_1))$$

Stationarity (translation invariance):

$$\mathrm{cov}(Z(\mathbf{x}_1, t_1), Z(\mathbf{x}_2, t_2)) = C(\mathbf{x}_1 - \mathbf{x}_2, t_1 - t_2)$$

Imbrication of the assumptions



From Gneiting, Genton & Guttorp (2007)

Three classes of covariance functions in $\mathbb{R}^d$

Class	Functions	Parameters
Stable	$C(\mathbf{h}) = b\,\exp(-(\theta\,\mathbf{h})^p)$	$b,\ \theta > 0; 0 < p \leq 2$
Whittle-Matérn (Bessel)	$C(\mathbf{h}) = b\,\frac{2^{1-\nu}}{\Gamma(\nu)}\,(\theta\,\mathbf{h})^\nu\,K_\nu(\theta\,\mathbf{h})$	$b,\ \theta,\ \nu > 0$
Cauchy	$C(\mathbf{h}) = b\,(1 + (\theta\,\mathbf{h})^p)^{-\nu}$	$b,\ \theta,\ \nu > 0; 0 < p \leq 2$

- Physically, the spatial and the time dimensions clearly play a distinct role, which should be reflected in the statistical model.

Gneiting's stationary space-time covariance functions

A continuous function $\varphi(r)$ with $r \geq 0$ is said to be *completely monotone,* if it possesses derivatives $\varphi^{(n)}$ of all orders and $(-1)^n \, \varphi^{(n)}(r) \geq 0$ for $r > 0$ and $n = 0, 1, 2, \ldots$

Theorem

Suppose that $\varphi(r)$*,* $r \geq 0$*, is a completely monotone function and that* $\psi(r)$*,* $r \geq 0$*, is a positive function with a completely monotone derivative. Then*

$$C(\mathbf{h}, u) = \frac{1}{\psi(u^2)^{d/2}} \, \varphi\left(\frac{|\mathbf{h}|^2}{\psi(u^2)}\right)$$

is a stationary covariance function on $\mathbb{R}^d \times \mathbb{R}$*.*

Gneiting's stationary space-time covariance functions

Example

The specific choices $\varphi(r) = b\ \exp(-a_1\, r^\gamma)$ and $\psi(r) = (1 + a_2\, r^\alpha)^\beta$ yield the parametric family of stationary space-time covariance functions

$$C(\mathbf{h}, u) = \frac{b}{(1 + a_2|u|^{2\alpha})^{\beta\, d/2}}\ \exp\left(-\frac{a_1|\mathbf{h}|^{2\gamma}}{(1 + a_2|u|^{2\alpha})^{\beta\,\gamma}}\right)$$

with smoothness parameters α, γ and
the **space-time interaction** parameter β taking values in $(0, 1]$.

- The purely spatial covariance function $C(\mathbf{h}, 0)$ is of the **stable covariance function class**,
- the purely temporal covariance function $C(0, u)$ belongs to the **Cauchy class**.

Frozen field model: non-symmetric covariance

- Geophysical processes often influenced by prevailing winds or ocean currents
- Idea of Lagrangian reference frame (moving with air or water mass)

Consider a spatial covariance C_S and a random velocity vector $\mathbf{V} \in \mathbb{R}^d$:

$$C(\mathbf{h}, u) = E[C_S(\mathbf{h} - \mathbf{V}u]$$

With prevailing winds we may consider a constant velocity vector $\mathbf{v}$ and the model is called the **frozen field** model.

Taylor's hypothesis

A stationary space-time covariance function C on $\mathbb{R}^d \times \mathbb{R}$ satistifies Taylor's hypothesis, if there exists a velocity vector $\mathbf{v} \in \mathbb{R}^d$ such that

$$C(0, u) = C(\mathbf{v}u, 0), \qquad u \in \mathbb{R}$$

- The covariance function of the frozen field model $C(\mathbf{h}, u) = C_S(\mathbf{h} - \mathbf{v}u)$ satisties Taylor's hypothesis.

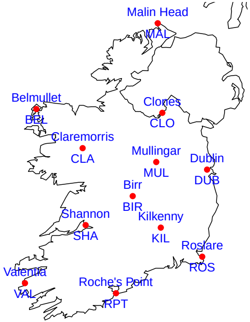
Irish wind case study

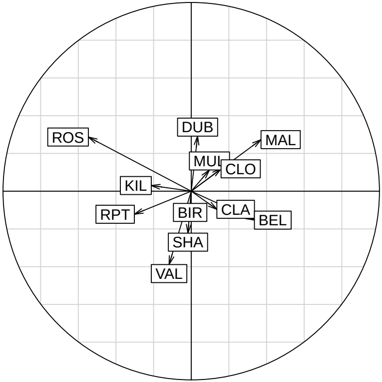
Gneiting, Genton & Guttorp (2007)

- Winds in Ireland are predominantly westerly, so that the velocity measures propagate from west to east.
- Temporal correlations lead or lag between W and E stations at a daily scale.
- Exploratory analysis shows a lack of full symmetry and thereby of separability in the correlation structure of the velocities.
- Fitting different parametric models: separable, fully symetric but not separable, stationary but not fully symmetric.
- Space-time simple kriging results show the best performance with the general stationary model in terms of four different performance measures.

Irish wind speed (daily, 1961-1978)

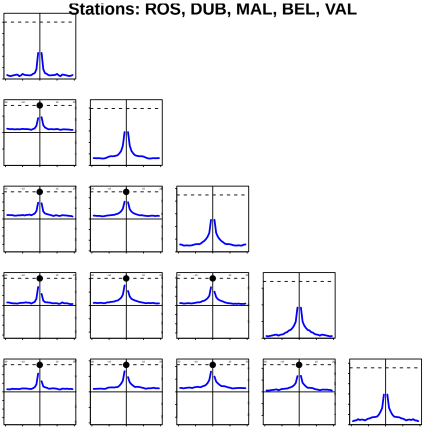
Haslett & Raftery (1989, with discussion)

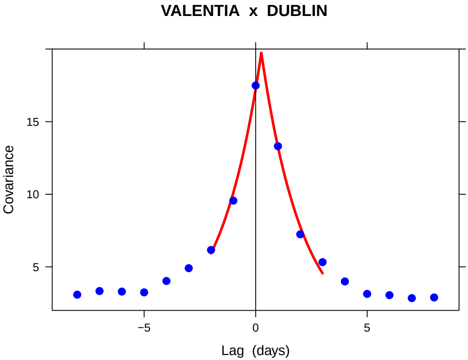




The correlation circle (PCA) reproduces the relative locations of stations on the geographical map.

Irish wind speed: cross-covariance functions





Asymmetric cross-covariances between E and W coastal stations.
The lagged correlation is about 6 hours.

Modelling a covariance function matrix with asymmetric cross-covariance functions

Li & Zhang (2011)

A simple and general approach to modelling asymmetric covariance functions for $\mathbf{h} \in \mathbb{R}^d$ is to introduce variable-specific vectors $\mathbf{a}_i$ and to use them to shift symmetric cross-covariance functions $C_{ij}(\mathbf{h})$:

$$C_{ij}^{a}(\mathbf{h}) = C_{ij}(\mathbf{h} + \mathbf{a}_i - \mathbf{a}_j)$$

thereby obtaining asymmetric cross-covariance functions $C_{ij}^{a}(\mathbf{h})$.

Perspectives: taper covariance models for multivariate localization

Roh et al (2015); Bevilacqua et al (2016)

- Univariate localization applied directly to multiple state variables may cause rank deficiency problems.
- Particular multivariate covariance functions can be used for multivariate tapering to replace the univariate tapering usually performed with Gaspari-Cohn functions.
- EnKF analysis can be improved at locations where some state variables are unobserved, when dealing more adequately with the cross-covariances through the multivariate tapering functions.

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