Multilevel ensemble Kalman filtering

Håkon Hoel[1] Kody Law[2] Raúl Tempone[3]

[1]Department of Mathematics, University of Oslo, Norway

[2]Oak Ridge National Laboratory, TN, USA

[3]Applied Mathematics and Computational Sciences, KAUST university, Saudi Arabia

11th International EnKF Workshop, Ulvik

Overview

Problem description

Consider the underlying and unobservable dynamics

$$u_{n+1} = \underbrace{u_n + \int_n^{n+1} a(u_t)\, dt + \int_n^{n+1} b(u_t)\, dW(t)}_{=:\Psi(u_n)}$$

with $u_n \in \mathbb{R}^d$, and Lipschitz continuous $a : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^{d\times\hat{d}}$.

And noisy observations

$$y_n = Hu_n + \gamma_n,$$

with i.i.d. $\gamma \sim N(0, \Gamma)$ and $H \in \mathbb{R}^{k\times d}$.

Objective: Let $Y_n := (y_1, y_2, \ldots, y_n)$ and let Y_n^{obs} be a sequence of *fixed* observations. Construct an efficient method for tracking $u_n|(Y_n = Y_n^{\mathrm{obs}})$. That is, approximate

$$\mathbb{E}\left[\phi(u_n)|Y_n = Y_n^{\mathrm{obs}}\right]$$

for an observable $\phi : \mathbb{R}^d \to \mathbb{R}$.

Abuse of notation: will write $u_n|Y_n^{\mathrm{obs}}$ to represent $u_n|(Y_n = Y_n^{\mathrm{obs}})$.

Problem description

Consider the underlying and unobservable dynamics

$$u_{n+1} = \underbrace{u_n + \int_n^{n+1} a(u_t)\, dt + \int_n^{n+1} b(u_t)\, dW(t)}_{=:\Psi(u_n)}$$

with $u_n \in \mathbb{R}^d$, and Lipschitz continuous $a : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^{d \times \hat{d}}$.

And noisy observations

$$y_n = Hu_n + \gamma_n,$$

with i.i.d. $\gamma \sim N(0, \Gamma)$ and $H \in \mathbb{R}^{k \times d}$.

Objective: Let $Y_n := (y_1, y_2, \ldots, y_n)$ and let Y_n^{obs} be a sequence of *fixed* observations. Construct an efficient method for tracking $u_n | (Y_n = Y_n^{\mathrm{obs}})$. That is, approximate

$$\mathbb{E}\left[\phi(u_n) | Y_n = Y_n^{\mathrm{obs}}\right]$$

for an observable $\phi : \mathbb{R}^d \to \mathbb{R}$.

Abuse of notation: will write $u_n | Y_n^{\mathrm{obs}}$ to represent $u_n | (Y_n = Y_n^{\mathrm{obs}})$.

Problem description

Consider the underlying and unobservable dynamics

$$u_{n+1} = \underbrace{u_n + \int_n^{n+1} a(u_t)\, dt + \int_n^{n+1} b(u_t)\, dW(t)}_{=:\Psi(u_n)}$$

with $u_n \in \mathbb{R}^d$, and Lipschitz continuous $a : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^{d \times \hat{d}}$.

And noisy observations

$$y_n = Hu_n + \gamma_n,$$

with i.i.d. $\gamma \sim N(0, \Gamma)$ and $H \in \mathbb{R}^{k \times d}$.

Objective: Let $Y_n := (y_1, y_2, \ldots, y_n)$ and let Y_n^{obs} be a sequence of *fixed* observations. Construct an efficient method for tracking $u_n | (Y_n = Y_n^{\mathrm{obs}})$. That is, approximate

$$\mathbb{E}\Big[\phi(u_n) | Y_n = Y_n^{\mathrm{obs}}\Big]$$

for an observable $\phi : \mathbb{R}^d \to \mathbb{R}$.

Abuse of notation: will write $u_n | Y_n^{\mathrm{obs}}$ to represent $u_n | (Y_n = Y_n^{\mathrm{obs}})$.

Overview

Ensemble Kalman Filtering

Predict

1. Compute (numerical solutions of) M particle paths one step forward

$$\widehat{v}_{n+1,i} = \Psi(v_{n,i}, \omega_i) \quad \text{for } i = 1, 2, \ldots, M.$$

2. Compute sample mean and covariance

$$\widehat{m}^{\mathrm{MC}}_{n+1} = E_M[\widehat{v}_{n+1}]$$
$$\widehat{C}^{\mathrm{MC}}_{n+1} = \mathrm{Cov}_M[\widehat{v}_{n+1}]$$

$$\text{where} \quad E_M[\widehat{v}_{n+1}] := \frac{1}{M}\sum_{i=1}^{M} \widehat{v}_{n+1,i}$$

$$\text{and} \quad \mathrm{Cov}_M[\widehat{v}_{n+1}] := E_M[\widehat{v}_{n+1}\widehat{v}^T_{n+1}] - E_M[\widehat{v}_{n+1}](E_M[\widehat{v}_{n+1}])^T.$$

Ensemble Kalman Filtering II

Update

1. Generate signal observations for the ensemble of particles

$$\tilde{y}_{n+1,i} = y^{\mathrm{obs}}_{n+1} + \gamma_{n+1,i} \quad \text{for } i = 1, 2 \ldots, M,$$

   with i.i.d. $\gamma_{n+1,1} \sim N(0, \Gamma)$.

2. Use signal observations to update particle paths

$$v_{n+1,i} = (I - K^{\mathrm{MC}}_{n+1} H)\widehat{v}_{n+1,i} + K^{\mathrm{MC}}_{n+1}\tilde{y}_{n+1,i},$$
$$\text{where} \quad K^{\mathrm{MC}}_{n+1} = \widehat{C}^{\mathrm{MC}}_{n+1} H^T (H \widehat{C}^{\mathrm{MC}}_{n+1} H^T + \Gamma)^{-1}.$$

Note: After the first step, all particles are correlated due to K^{MC}_{n+1}.

From EnKF to mean field EnKF

For studying convergence properties of EnKF it is useful to introduce the mean field EnKF (MFEnKF)

$$\text{Pr}\begin{cases} \widehat{v}^{\text{MF}}_{n+1,i} &= \Psi(v^{\text{MF}}_{n,i}, \omega_i) \\ \widehat{m}^{\text{MF}}_{n+1} &= \mathbb{E}\left[\widehat{v}^{\text{MF}}_{n+1,i}\right] \\ \widehat{C}^{\text{MF}}_{n+1} &= \text{Cov}[\widehat{v}^{\text{MF}}_{n+1,i}], \end{cases} \quad \text{Up}\begin{cases} K^{\text{MF}}_{n+1} &= \widehat{C}^{\text{MF}}_{n+1} H^T (H \widehat{C}^{\text{MF}}_{n+1} H^T + \Gamma)^{-1} \\ \tilde{y}_{n+1,i} &= y^{\text{obs}}_{n+1} + \gamma_{n+1,i} \\ v^{\text{MF}}_{n+1,i} &= (I - K^{\text{MF}}_{n+1} H) v^{\text{MF}}_{n+1,i} + K^{\text{MF}}_{n+1} \tilde{y}_{n+1,i}. \end{cases}$$

and in comparison, EnKF

$$\text{Pr}\begin{cases} \widehat{v}_{n+1,i} &= \Psi(v_{n,i}, \omega_i) \\ \widehat{m}^{\text{MC}}_{n+1} &= E_M[\widehat{v}_{n+1}] \\ \widehat{C}^{\text{MC}}_{n+1} &= \text{Cov}_M[\widehat{v}_{n+1}] \end{cases} \quad \text{Up}\begin{cases} K^{\text{MC}}_{n+1} &= \widehat{C}^{\text{MC}}_{n+1} H^T (H \widehat{C}^{\text{MC}}_{n+1} H^T + \Gamma)^{-1} \\ \tilde{y}_{n+1,i} &= y^{\text{obs}}_{n+1} + \gamma_{n+1,i} \\ v_{n+1,i} &= (I - K^{\text{MC}}_{n+1} H) \widehat{v}_{n+1,i} + K^{\text{MC}}_{n+1} \tilde{y}_{n+1,i}. \end{cases}$$

- When underlying dynamics is linear with Gaussian additive noise and u_0 Gaussian, it holds that $\mu^{\text{MF}}_n(dx) = \mathbb{P}\left(u_n \in dx | Y^{\text{obs}}_n\right)$, where $\mu^{\text{MF}}_n = \text{Law}(v^{\text{MF}}_{n,i})$.
- In nonlinear settings, we use as approximation goal

$$\int_{\mathbb{R}^d} \phi(x) \mu^{\text{MF}}_n(dx). \quad \text{NB!}(\mu^{\text{MF}}_n \neq \mathbb{P}\left(u_n \in \cdot | Y^{\text{obs}}_n\right).)$$

From EnKF to mean field EnKF

For studying convergence properties of EnKF it is useful to introduce the mean field EnKF (MFEnKF)

$$\text{Pr}\begin{cases} \widehat{v}^{\mathrm{MF}}_{n+1,i} &= \Psi(v^{\mathrm{MF}}_{n,i}, \omega_i) \\ \widehat{m}^{\mathrm{MF}}_{n+1} &= \mathbb{E}\left[\widehat{v}^{\mathrm{MF}}_{n+1,i}\right] \\ \widehat{C}^{\mathrm{MF}}_{n+1} &= \mathrm{Cov}[\widehat{v}^{\mathrm{MF}}_{n+1,i}], \end{cases} \quad \text{Up}\begin{cases} K^{\mathrm{MF}}_{n+1} &= \widehat{C}^{\mathrm{MF}}_{n+1} H^T (H\widehat{C}^{\mathrm{MF}}_{n+1} H^T + \Gamma)^{-1} \\ \tilde{y}_{n+1,i} &= y^{\mathrm{obs}}_{n+1} + \gamma_{n+1,i} \\ v^{\mathrm{MF}}_{n+1,i} &= (I - K^{\mathrm{MF}}_{n+1} H) v^{\mathrm{MF}}_{n+1,i} + K^{\mathrm{MF}}_{n+1}\tilde{y}_{n+1,i}. \end{cases}$$

and in comparison, EnKF

$$\text{Pr}\begin{cases} \widehat{v}_{n+1,i} &= \Psi(v_{n,i}, \omega_i) \\ \widehat{m}^{\mathrm{MC}}_{n+1} &= E_M[\widehat{v}_{n+1}] \\ \widehat{C}^{\mathrm{MC}}_{n+1} &= \mathrm{Cov}_M[\widehat{v}_{n+1}] \end{cases} \quad \text{Up}\begin{cases} K^{\mathrm{MC}}_{n+1} &= \widehat{C}^{\mathrm{MC}}_{n+1} H^T (H\widehat{C}^{\mathrm{MC}}_{n+1} H^T + \Gamma)^{-1} \\ \tilde{y}_{n+1,i} &= y^{\mathrm{obs}}_{n+1} + \gamma_{n+1,i} \\ v_{n+1,i} &= (I - K^{\mathrm{MC}}_{n+1} H) \widehat{v}_{n+1,i} + K^{\mathrm{MC}}_{n+1}\tilde{y}_{n+1,i}. \end{cases}$$

- When underlying dynamics is linear with Gaussian additive noise and u_0 Gaussian, it holds that $\mu_n^{\mathrm{MF}}(dx) = \mathbb{P}\left(u_n \in dx | Y_n^{\mathrm{obs}}\right)$, where $\mu_n^{\mathrm{MF}} = \mathrm{Law}(v^{\mathrm{MF}}_{n,i})$.
- In nonlinear settings, we use as approximation goal

$$\int_{\mathbb{R}^d} \phi(x)\mu_n^{\mathrm{MF}}(dx). \quad \text{NB!}(\mu_n^{\mathrm{MF}} \neq \mathbb{P}\left(u_n \in \cdot | Y_n^{\mathrm{obs}}\right).)$$

Convergence of EnKF

Theorem 1 (Le Gland et al. (2009))

Consider the dynamics and observations,

$$\begin{aligned} u_{n+1} &= f(u_n) + \xi_{n+1}, \quad \xi_{n+1} \sim N(0, \Sigma), \\ y_{n+1} &= Hu_{n+1} + \gamma_{n+1}, \quad \gamma_{n+1} \sim N(0, \Gamma), \end{aligned}$$

and assume $u_0 \in L^p(\Omega)$ *for any* $p \geq 1$, *and that*

$$\max(|f(x) - f(x')|, |\phi(x) - \phi(x')|) \leq C|x - x'|(1 + |x|^s + |x'|^s), \text{ for an } s \geq 0.$$

Then, for the EnKF update ensemble $\{v_{n,i}\}_{i=1}^M$,

$$\sup_{M \geq 1} \sqrt{M} \left(\mathbb{E}\left[\left| \sum_{i=1}^M \frac{\phi(v_{n,i})}{M} - \int_{\mathbb{R}^d} \phi(x) \mu_n^{\mathrm{MF}}(dx) \right|^p \right] \right)^{1/p} < \infty.$$

for any order $p \geq 1$ *and finite* n.

Extension to further nonlinear settings in [Law et al. (2014)].

Computational cost of EnKF

- To meet the constraint

$$\left(\mathbb{E}\left[\left| \sum_{i=1}^{M} \frac{\phi(v_{n,i})}{M} - \int_{\mathbb{R}^d} \phi(x) \mu_n^{\mathrm{MF}}(dx) \right|^p \right] \right)^{1/p} = \mathcal{O}(\epsilon),$$

  one thus needs ensemble of size $M = \mathcal{O}(\epsilon^{-2})$.

- How does the computational cost increase if the EnKF dynamics has to be sampled using a numerical solver for which $|\mathbb{E}[\Psi_{\Delta t} - \Psi]| = \mathcal{O}(\Delta t^{\alpha})$?
- Short answer (under additional assumptions): the cost increases to $\mathcal{O}(\epsilon^{-(2+1/\alpha)})$.

Overview

Multilevel EnKF (MLEnKF)

Prediction

- Compute an ensemble of particle paths on a hierarchy of accuracy levels

$$\widehat{v}^{\ell-1}_{n+1,i} = \Psi^{\ell-1}(v^{\ell-1}_{n,i}, \omega_{\ell,i}), \quad \widehat{v}^{\ell}_{n+1,i} = \Psi^{\ell}(v^{\ell}_{n,i}, \omega_{\ell,i}),$$

for the levels $\ell = 0, 1, \ldots, L$ and $i = 1, 2, \ldots, M_\ell$.

- Multilevel approximation of mean and covariance matrices:

$$\widehat{m}^{\mathrm{ML}}_{n+1} = \sum_{\ell=0}^{L} E_{M_\ell}[\widehat{v}^{\ell}_{n+1} - \widehat{v}^{\ell-1}_{n+1}],$$

$$\widehat{C}^{\mathrm{ML}}_{n+1} = \sum_{\ell=0}^{L} \mathrm{Cov}_{M_\ell}[\widehat{v}^{\ell}_{n+1}] - \mathrm{Cov}_{M_\ell}[\widehat{v}^{\ell-1}_{n+1}]$$

Notice the telescoping properties $\mathbb{E}\left[\widehat{m}^{\mathrm{ML}}_{n+1}\right] = \mathbb{E}\left[\widehat{v}^{L}_{n+1}\right]$ and $\mathbb{E}\left[\widehat{C}^{\mathrm{ML}}_{n+1}\right] = \mathrm{Cov}(\widehat{v}^{L}_{n+1}) + O(1/M_L)$.

MLEnKF update step

Update

For $\ell = 0, 1, \dots, L$ and $i = 1, 2, \dots, M_\ell$,

$$\begin{aligned}
\tilde{y}^{\ell}_{n+1,i} &= y^{\mathrm{obs}}_{n+1} + \gamma^{\ell}_{n+1,i}, \qquad \text{i.i.d. } \gamma^{\ell}_{n+1,i} \sim N(0, \Gamma) \\
v^{\ell-1}_{n+1,i} &= (I - K^{\mathrm{ML}}_{n+1} H)\widehat{v}^{\ell-1}_{n+1,i} + K^{\mathrm{ML}}_{n+1}\tilde{y}^{\ell}_{n+1,i}, \\
v^{\ell}_{n+1,i} &= (I - K^{\mathrm{ML}}_{n+1} H)\widehat{v}^{\ell}_{n+1,i} + K^{\mathrm{ML}}_{n+1}\tilde{y}^{\ell}_{n+1,i},
\end{aligned}$$

$$\text{where} \quad K^{\mathrm{ML}}_{n+1} = \widehat{C}^{\mathrm{ML}}_{n+1} H^{\mathsf{T}} (H \widehat{C}^{\mathrm{ML}}_{n+1} H^{\mathsf{T}} + \Gamma)^{-1}.$$

Convergence of MLEnKF

For observables $\phi : \mathbb{R}^d \to \mathbb{R}$, introduce notation

$$\mu_n^{\mathrm{ML}}(\phi) := \sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} \phi(v_{n,i}^{\ell}) - \phi(v_{n,i}^{\ell-1}).$$

and

$$\mu_n^{\mathrm{MF}}(\phi) := \int_{\mathbb{R}^d} \phi(x) \mu_n^{\mathrm{MF}}(dx).$$

Question: Under what assumptions and at what cost can one achieve

$$\|\mu_n^{\mathrm{ML}}(\phi) - \mu_n^{\mathrm{MF}}(\phi)\|_{L^p(\Omega)} = \mathcal{O}(\epsilon)?$$

Assumption 1

Consider the dynamics

$$u_{n+1} = \Psi(u_n) = u_n + \int_n^{n+1} a(u_t)dt + \int_n^{n+1} b(u_t)dW(t), \quad n = 0, 1, \ldots$$

with $u_0 \in \cap_{p\in\mathbb{N}} L^p(\Omega)$ *and a hierarchy of numerical solvers* $\{\Psi^\ell\}_{\ell=0}^\infty$. *Furthermore, assume the observable* $\phi : \mathbb{R}^d \to \mathbb{R}$ *satisfies*

$$|\phi(x) - \phi(x')| \leq C|x - x'|(1 + |x|^s + |x'|^s), \text{ for an } s \geq 0,$$

that there exists positive constants $\alpha, \beta > 0$, *and an positive exponentially increasing sequence* $\{N_\ell\}_\ell$ *such that for all* $u, v \in \cap_{p\in\mathbb{N}} L^p(\Omega)$,

(i) $\left|\mathbb{E}\left[\phi(\Psi^\ell(u)) - \phi(\Psi(v))\right]\right| \lesssim N_\ell^{-\alpha}$, *provided that* $|\mathbb{E}[u - v]| \lesssim N_\ell^{-\alpha}$,

(ii) $\|\phi(\Psi^\ell(v)) - \phi(\Psi^{\ell-1}(v))\|_p \lesssim N_\ell^{-\beta}$, *for all* $p \geq 1$,

(iii) $\mathrm{Cost}\left(\Psi^\ell(v)\right) \lesssim N_\ell$.

Theorem 2 (MLEnKF accuracy vs. cost)

Suppose Assumption 1 holds. Then, for any $\epsilon > 0$ *and* $p \geq 2$, *there exists an* $L > 0$ *and* $\{M_\ell\}_{\ell=0}^{L}$ *such that*

$$\|\mu_n^{\mathrm{ML}}(\phi) - \mu_n^{\mathrm{MF}}(\phi)\|_p \lesssim \epsilon.$$

And

$$\mathrm{Cost}\,(\mathrm{MLEnKF}) \lesssim \begin{cases} \left(|\log(\epsilon)|^{1-n}\epsilon\right)^{-2}, & \text{if} \quad \beta > 1, \\ \left(|\log(\epsilon)|^{1-n}\epsilon\right)^{-2} |\log(\epsilon)|^3, & \text{if} \quad \beta = 1, \\ \left(|\log(\epsilon)|^{1-n}\epsilon\right)^{-\left(2+\frac{1-\beta}{\alpha}\right)}, & \text{if} \quad \beta < 1. \end{cases} \qquad (1)$$

In comparison,

$$\|\mu_n^{\mathrm{EnKF}}(\phi) - \mu_n^{\mathrm{MF}}(\phi)\|_p \lesssim \epsilon,$$

is achieved at cost $\mathcal{O}\left(\epsilon^{-\left(2+\frac{1}{\alpha}\right)}\right)$.

Theorem 2 (MLEnKF accuracy vs. cost)

Suppose Assumption 1 holds. Then, for any $\epsilon > 0$ *and* $p \geq 2$*, there exists an* $L > 0$ *and* $\{M_\ell\}_{\ell=0}^{L}$ *such that*

$$\|\mu_n^{\mathrm{ML}}(\phi) - \mu_n^{\mathrm{MF}}(\phi)\|_p \lesssim \epsilon.$$

And

$$\mathrm{Cost}\,(\mathrm{MLEnKF}) \lesssim \begin{cases} (|\log(\epsilon)|^{1-n}\epsilon)^{-2}, & \textit{if} \quad \beta > 1, \\ (|\log(\epsilon)|^{1-n}\epsilon)^{-2}\,|\log(\epsilon)|^3\,, & \textit{if} \quad \beta = 1, \\ (|\log(\epsilon)|^{1-n}\epsilon)^{-\left(2+\frac{1-\beta}{\alpha}\right)}, & \textit{if} \quad \beta < 1. \end{cases} \quad (1)$$

In comparison,

$$\|\mu_n^{\mathrm{EnKF}}(\phi) - \mu_n^{\mathrm{MF}}(\phi)\|_p \lesssim \epsilon,$$

is achieved at cost $\mathcal{O}\left(\epsilon^{-\left(2+\frac{1}{\alpha}\right)}\right)$.

Central idea in the proof

Introduce

$$\mu_n^{\mathrm{MLMF}}(\phi) := \sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} \phi(v_n^{\mathrm{MF},\ell}(\omega_{i,\ell})) - \phi(v_n^{\mathrm{MF},\ell-1}(\omega_{i,\ell}))$$

$$\mu_n^{\mathrm{MF,L}}(\phi) := \mathbb{E}\left[\phi(v_n^{\mathrm{MF,L}})\right],$$

and bound MLEnKF error by

$$\begin{aligned}
\|\mu_n^{\mathrm{ML}}(\phi) - \mu_n^{\mathrm{MF}}(\phi)\|_p &\leq \|\mu_n^{\mathrm{ML}}(\phi) - \mu_n^{\mathrm{MLMF}}(\phi)\|_p \\
&\quad + \|\mu_n^{\mathrm{MLMF}}(\phi) - \mu_n^{\mathrm{MF,L}}(\phi)\|_p + \|\mu_n^{\mathrm{MF,L}}(\phi) - \mu_n^{\mathrm{MF}}(\phi)\|_p \\
&\leq c \sum_{\ell=0}^{L} \left[\|v_n^{\ell} - v_n^{\mathrm{MF},\ell}\|_{\hat{p}} + \frac{\|v_n^{\mathrm{MF},\ell} - v_n^{\mathrm{MF},\ell-1}\|_{\hat{p}}}{M_\ell^{1/2}} \right] + \left|\mathbb{E}\left[\phi(v_n^{\mathrm{MF,L}}) - \phi(v_n^{\mathrm{MF}})\right]\right| \\
&\leq c \left(\epsilon + \sum_{\ell=0}^{L} M_\ell^{-1/2} N_\ell^{-\beta/2} + N_L^{-\alpha} \right)
\end{aligned}$$

Overview

Numerical example

Underlying dynamics is the Ornstein–Uhlenbeck SDE

$$du = -udt + 0.5dW(t),$$

with a set of observations

$$y_n = u_n + \gamma_n, \qquad i.i.d.\ \gamma_n \sim N(0, 0.04)$$

Solvers: Hierarchy of Milstein solution operators $\{\Psi_\ell\}_{\ell=0}^{L}$ with $\Delta t^\ell = \mathcal{O}\left(2^{-\ell}\right)$.

Compare the approximation errors for the observable $\phi(x) = x$ in terms of the RMSE

$$\sqrt{\sum_{n=1}^{N} \frac{|\mu_n^{\mathrm{ML}}(\phi) - \mu_n^{\mathrm{MF}}(\phi)|^2}{N}}.$$

Numerical example

Underlying dynamics is the Ornstein–Uhlenbeck SDE

$$du = -udt + 0.5dW(t),$$

with a set of observations

$$y_n = u_n + \gamma_n, \qquad i.i.d.\ \gamma_n \sim N(0, 0.04)$$

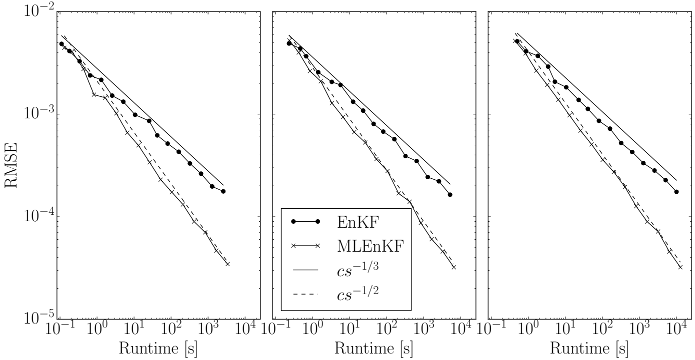
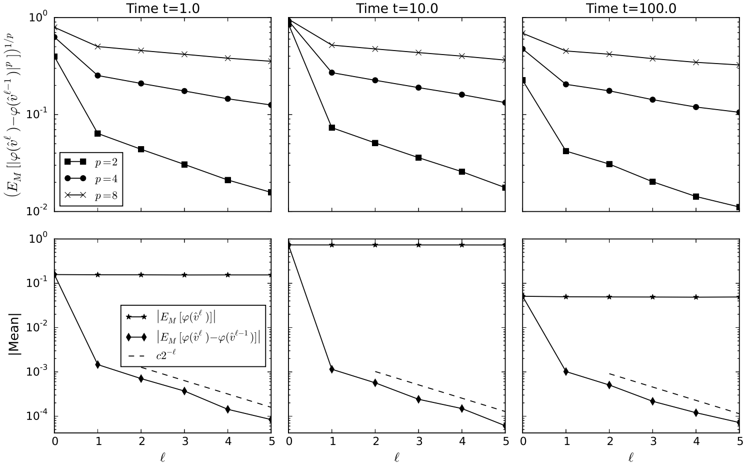


Figure: From left to right: $N = 100, 200$ and 400.

OU example

Consider less regular observable $\phi(x) := \mathbf{1}\{x > 0.1\}$. Outside the scope of our theory since it does not hold that

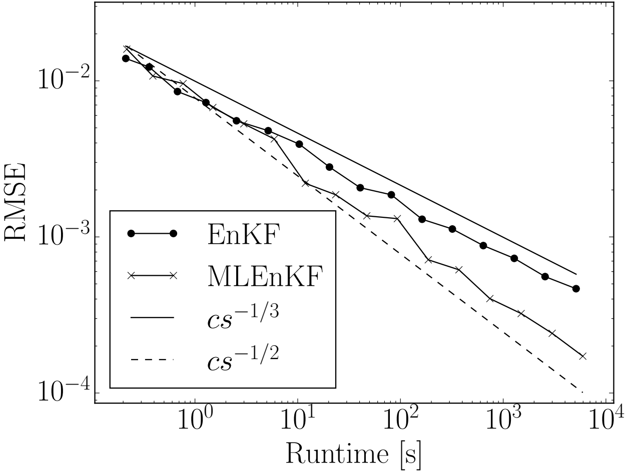
$$\|\phi(\Psi^{\ell}(v)) - \phi(\Psi^{\ell-1}(v))\|_p \lesssim N_{\ell}^{-\beta}, \quad \forall p \geq 2.$$



OU example

Consider less regular observable $\phi(x) := \mathbf{1}\{x > 0.1\}$. Outside the scope of our theory since it does not hold that

$$\|\phi(\Psi^{\ell}(v)) - \phi(\Psi^{\ell-1}(v))\|_p \lesssim N_{\ell}^{-\beta}, \quad \forall p \geq 2.$$



Overview

Extension of MLEnKF to infinite dimensional state spaces

- Work in progress with Alexey Chernov, Kody Law, Fabio Nobile and Tempone.
- Infinite dimensional stochastic dynamics:

$$u_{n+1} = \Psi(u_n)$$

where $u_n \in L^p(\Omega; \mathcal{H})$ with $\mathcal{H} = \mathrm{Span}(\{\nu_i\}_{i=1}^{\infty})$, and $\Psi : L^p(\Omega; \mathcal{H}) \to L^p(\Omega; \mathcal{H})$.

- And finite dimensional observations

$$y_n = Hu_n + \gamma_n,$$

with linear $H : \mathcal{H} \to \mathbb{R}^m$

- Introduce nested hierarchy of Hilbert spaces

$$\mathcal{H}_0 \subset \mathcal{H}_1 \subset \ldots \subset \mathcal{H}_\infty = \mathcal{H},$$

where $\mathcal{H}_\ell = \mathrm{Span}(\{\nu_i\}_{i=1}^{N_\ell})$ and work with a hierarchy of solvers

$$\Psi^\ell : L^p(\Omega; \mathcal{H}_\ell) \to L^p(\Omega; \mathcal{H}_\ell).$$

Conclusion

- Extended EnKF to multilevel EnKF.
- Verified asymptotic efficiency gain for approximations of expectation of observables. We hope to improve result further!
- Further extension of MLEnKF to infinite dimensional state space is work in progress.

References

1 Håkon Hoel, Kody JH Law, and Raul Tempone, *Multilevel ensemble Kalman filtering*, SIAM J. Numer. Anal., 54(3), pp. 18131839, (2016).

2 François Le Gland, Valérie Monbet, Vu-Duc Tran, et al., *Large sample asymptotics for the ensemble kalman filter*, The Oxford Handbook of Nonlinear Filtering, (2011), pp. 598–631.

3 Kody JH Law, Hamidou Tembine, and Raul Tempone, *Deterministic methods for nonlinear filtering, part i: Mean-field ensemble kalman filtering*, arXiv preprint arXiv:1409.0628, (2014).