Geostatistical change-of-support models

for irregular grids

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Change of support: the decisional problem

Sampling cm^3 and extracting m^3: the selection is performed on large volumes (blocks)

Exploration:
samples cm^3

→

Extraction:
blocks m^3

Decision

Processing

→

Metal

Dump

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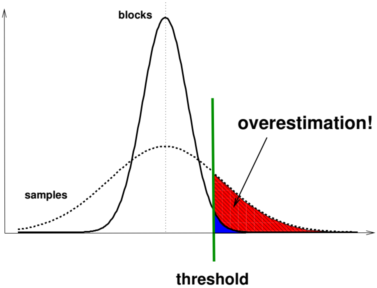
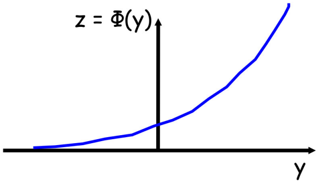


Anamorphosis

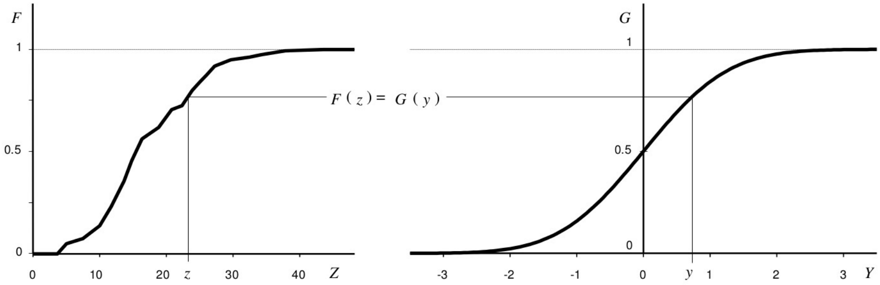
Gaussian anamorphosis

- The variable of interest is rarely Gaussian
- A Gaussian transformation (anamorphosis) is introduced:
  - $Z = \varphi(Y)$, with Y standard normal
  - Z increases with Y
  - φ is a strictly increasing function of Y



Normal score transform

- Distribution functions



Gaussian model : Hermite polynomials

- Development of the Gaussian anamorphosis function or of any other function of a Gaussian variable into Hermite polynomials.
- The Hermite orthogonal polynomials go with the normal distribution.
- Knowing the Gaussian density function $g(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{y^2}{2})$

$$\begin{aligned} H_k(y) &= \frac{1}{\sqrt{k!}} \frac{k^{\text{th}} \text{ derivative of Gaussian density}}{\text{Gaussian density}} \\ &= \frac{1}{\sqrt{k!}} \frac{g^{(k)}(y)}{g(y)} \qquad \text{with} \quad k = 0, 1, \ldots \end{aligned}$$

which yields:

$$H_0(y) = 1 \qquad H_1(y) = -y \qquad H_2(y) = \frac{y^2 - 1}{\sqrt{2}} \qquad H_3(y) = \frac{-y^3 + 3y}{\sqrt{6}}$$

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Development of a fonction into Hermite polynomials

- Any (square integrable) function $f(Y)$ can be developed into Hermite polynomials:

$$f(Y) = \sum_{k\geq 0} f_k\, H_k(Y)$$

knowing the coefficients f_k are given by

$$E[f(Y)\, H_l(Y)] = \sum_{k\geq 0} f_k\, E[H_k(Y)\, H_l(Y)] = f_l$$

In particular:

- $f_0 = E[f(Y)]$
- $\mathrm{var}(f(Y)) = \sum_{k\geq 1} (f_k)^2$

Spatial bivariate Gaussian model

- For $(Y(\mathbf{x}), Y(\mathbf{x}+\mathbf{h}))$ bivariate normal with correlation function $\rho(\mathbf{h})$ we have

$$\begin{aligned} \text{for } k > 0: \quad & \text{cov}(\, H_k(Y(\mathbf{x})), H_k(Y(\mathbf{x}+\mathbf{h}))\,) = & (\rho(\mathbf{h}))^k \\ \text{for } k \neq l: \quad & \text{cov}(\, H_l(Y(\mathbf{x})), H_k(Y(\mathbf{x}+\mathbf{h}))\,) = & 0 \end{aligned}$$

- $H_k(Y(\mathbf{x}))$ is more weakly auto-correlated with increasing k
- There is neither cross nor direct spatial correlation between two polynomials of different degree.

Discrete Gaussian change-of-support model (DGM)

Discretized Gaussian change-of-support model (DGM)

Matheron (1976)

$\underline{\mathbf{x}}$ is a point at a random position within the block v.
By Cartier's relation:

$$E[Z(\underline{\mathbf{x}}) \mid Z(v)] = Z(v)$$

A Gaussian anamorphosis is computed from the station data:

$$Z(\mathbf{x}) = \varphi(Y(\mathbf{x})) = \sum_{k=0}^{\infty} \varphi_k H_k(Y(\mathbf{x}))$$

where H_k are Hermite polynomials. Applying Cartier:

$$E[Z(\underline{\mathbf{x}}) \mid Z(v)] = E[\varphi(Y(\underline{\mathbf{x}})) \mid \varphi(Y(v))] = \varphi_v(Y(v)) = \sum_{k=0}^{\infty} \varphi_k \, r^k \, H_k(Y(v))$$

where r is the **point-block** coefficient, $0 < r \leq 1$.

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The change-of-support coefficient r

The **block variance** of $Z(v)$ is computed from the **point variogram** $\gamma(\mathbf{h})$ (or the covariance function $C(\mathbf{h})$):

$$\mathrm{var}(Z(v)) \quad = \quad \frac{1}{|v|^2} \int_v \int_v C(\mathbf{x}-\mathbf{x}')\, d\mathbf{x} d\mathbf{x}'$$

The block variance can be expressed in terms of the block anamorphosis:

$$\mathrm{var}(Z(v)) = \mathrm{var}(\varphi_v(Y(v))) = \sum_{k=1}^{\infty} \varphi_k\, r^{2k}$$

The **point-block coefficient** r is obtained by inverting this relation.

Change of support: geostatistical simulation

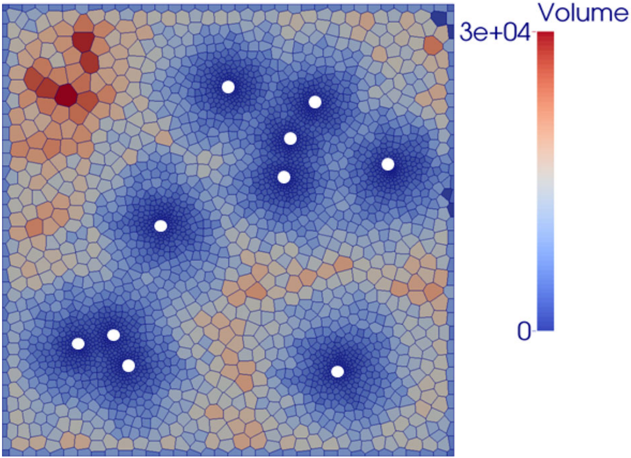
Change-of-support by upscaling point simulation

- On a **regular grid** upscaled values can be obtained without an explicit change-of-support model:
  1. point values are generated by geostatistical simulation on a fine-scale grid;
  2. upscaled values are obtained by averaging the point values on the block support.

- For **unstructured grids** as used in reservoir models of the petroleum industry:
  - the creation and storage of a fine-scale regular grid may be too time-demanding (very different cell volumes);
  - artifacts may appear if the chosen refinement was not sufficient.

- For geostatistical simulation on unstructured grids an **explicit change-of-support model** is recommended.

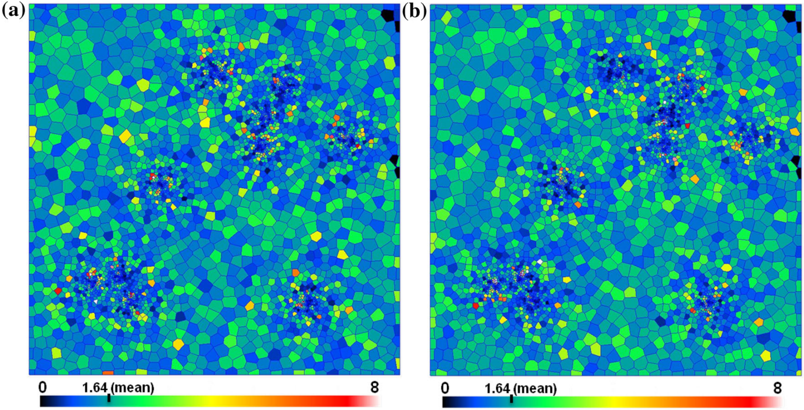
Unstructured grid: very different cell volumes



The white points indicate oil well locations.

Unconditional simulation example

Zaytsev et al. (2015)



(a) Upscaled point unconditional simulation.

(b) Simulation with DGM.

A variant of
the Discrete Gaussian Model

Discrete Gaussian models DGM1 and DGM2

- In the original (**DGM1**) model proposed by Matheron (1976) the point-block coefficient r is computed from the relation:

$$\sum_{k=1}^{\infty} \varphi_k \, r^{2k} \quad = \quad \frac{1}{|v|^2} \int_v \int_v C(\mathbf{x}-\mathbf{x}') \, d\mathbf{x} d\mathbf{x}'$$

  where the right-hand side represents the variance of $Z(\mathbf{x})$.

- Emery (2007) proposes a variant (**DGM2**) in which the point-block coefficient r is computable directly from the variance of $Y(\mathbf{x})$:

$$r \quad = \quad \frac{1}{|v|^2} \int_v \int_v \rho(\mathbf{x}-\mathbf{x}') \, d\mathbf{x} d\mathbf{x}'$$

  which seems attractive in applications.

- DGM2 is based on the assumption that the bivariate distribution of a pair $(Y(\underline{x}), Y(\underline{x}'))$ with $\underline{x}, \underline{x}' \in v$ is bi-Gaussian, while Matheron (1976) showed that the distribution of the pair actually is bi-Hermitian.

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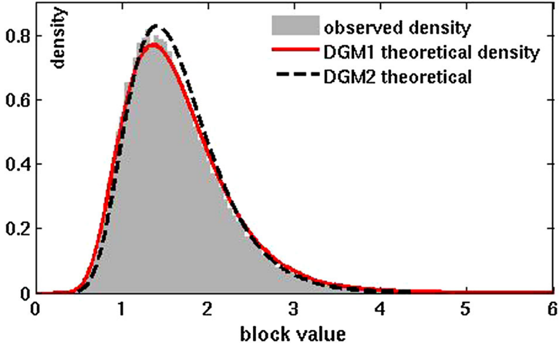
Statistical analysis of simulation results

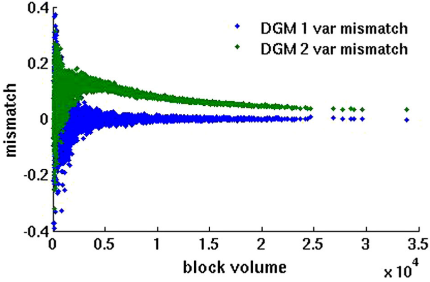
- Statistics were computed based on 50 000 unconditional simulations.



- Graphical comparison of densities for largest block in grid.
- For small blocks the densities are undistinguishable.

DGM1 vs DGM2

- Graphical representation of the mismatch between practical and theoretical block variance depending on block size.



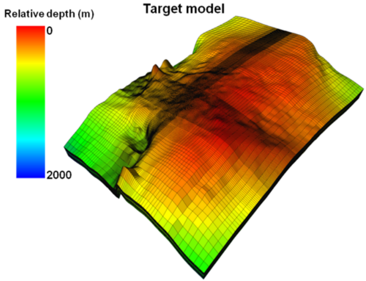
- The mismatch for DGM1 (blue) values is unbiased. For DGM2 (green) they reveal a bias.
- But: the relative value of bias does not exceed 5% of the variance of $Z(\mathbf{x})$.
- If the range of the covariance function is large as compared to block size (a frequent situation in practice), the bias is negligible.

Simulation of porosity on an offshore field

(Zaytsev et al. 2015)

Simulation of porosity

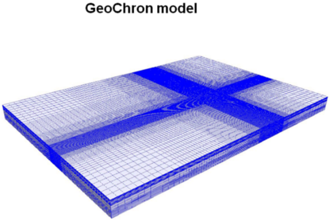
Tartan meshed offshore gas field model



Original model

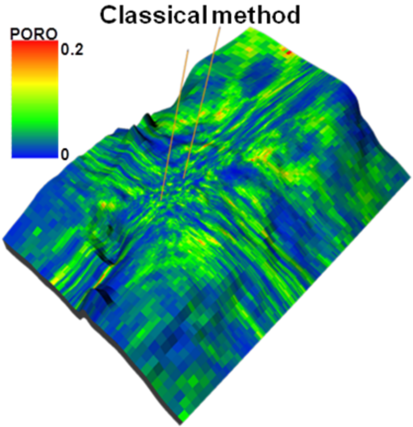
$\Longrightarrow$

Depositional space

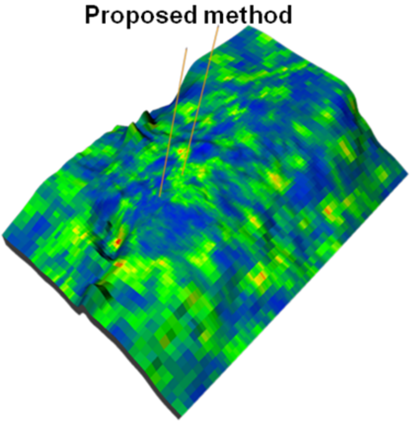


GeoChron model

Change of support: porosity



Averaging point simulation on original grid



Using DGM on GeoChron grid

Conclusion

- Explicit geostatistical change-of-support models are useful to anticipate the deformation of statistical parameters as a function of support in the presence of auto-correlation.
- The discrete Gaussian model can be applied to data with different, non-point supports (Brown et al. 2008).
- Non-Gaussian (isofactorial) models are available.

Acknowledgements

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