On the ensemble Rauch-Tung-Striebel smoother (EnRTS) and its equivalence to the ensemble Kalman smoother (EnKS)

Patrick Nima Raanes

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11th EnKF workshop, Ulvik, June 19, 2016

UNIVERSITY OF OXFORD

NERSC

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EnRTS vs. EnKS

Benchmarks (from 2013 workshop):

EnRTS	0.43	0.24	0.15	0.16
EnKS	0.49	0.23	0.16	0.14

$\rightarrow N \rightarrow$

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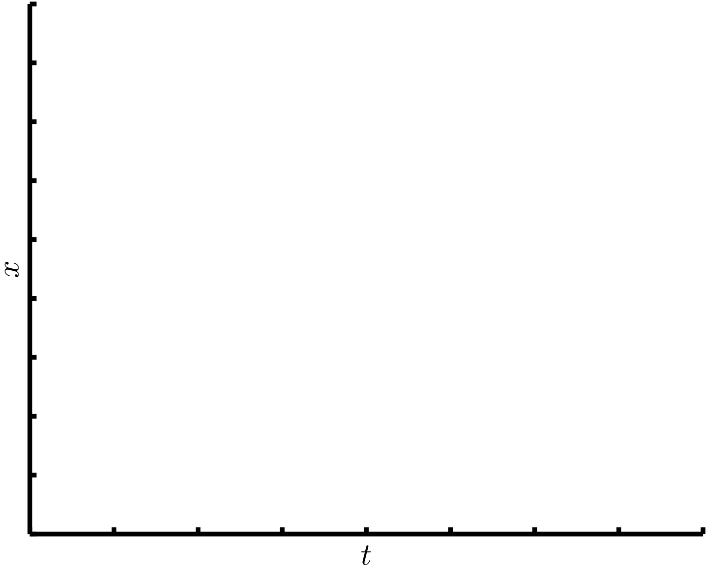

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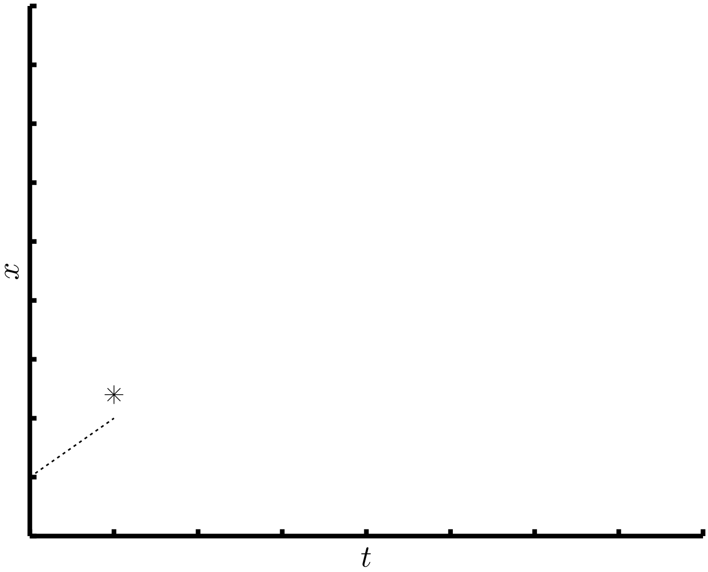

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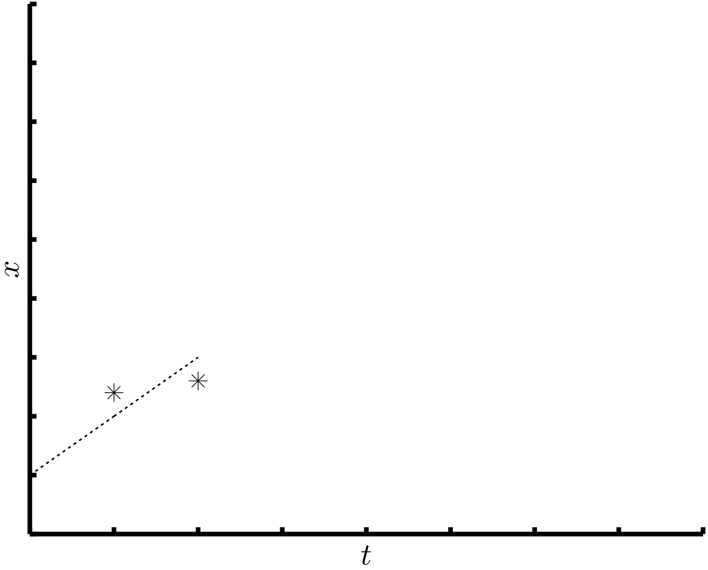

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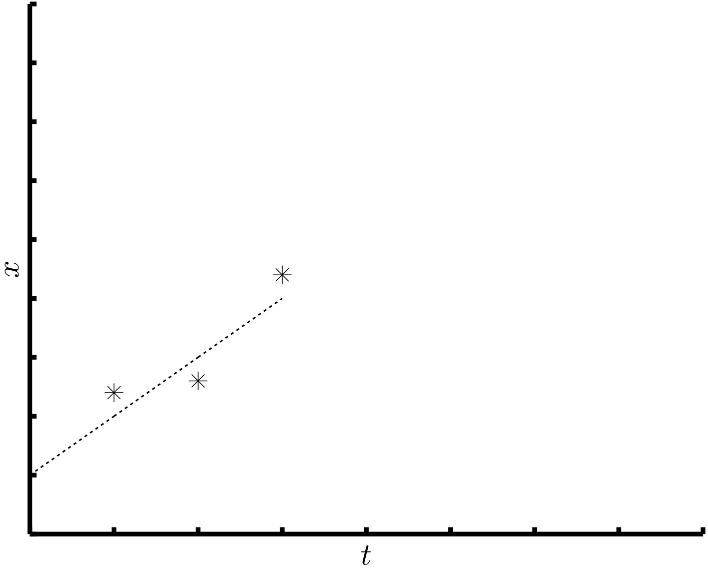

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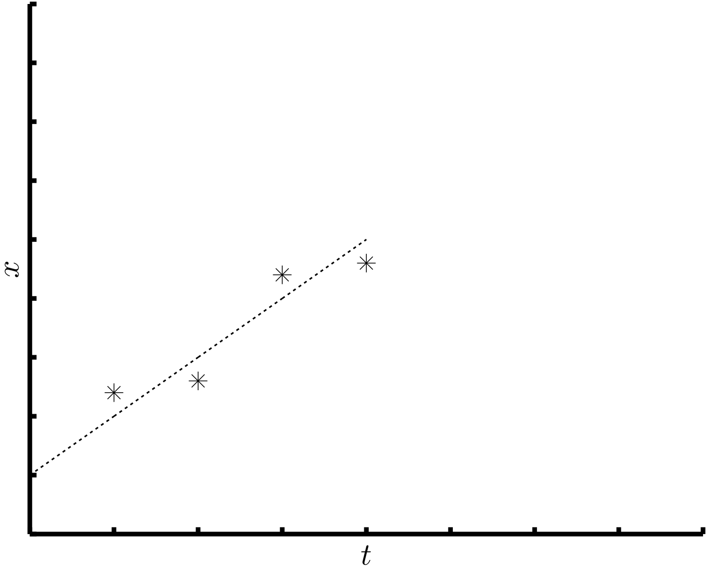

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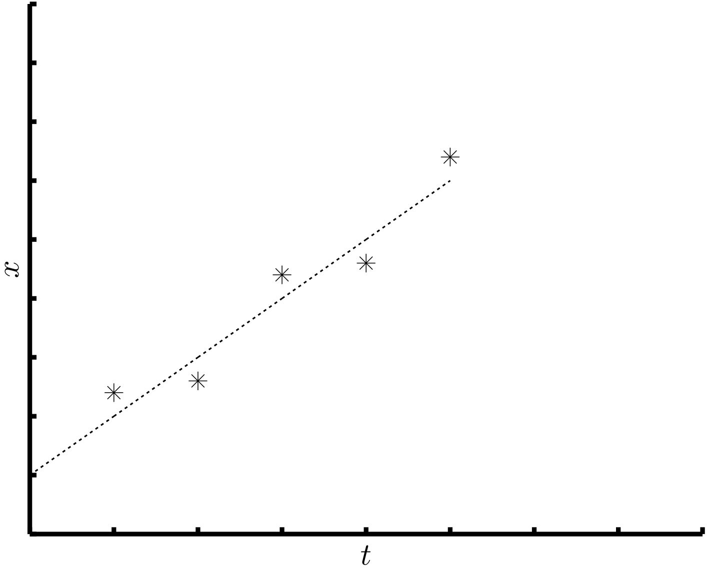

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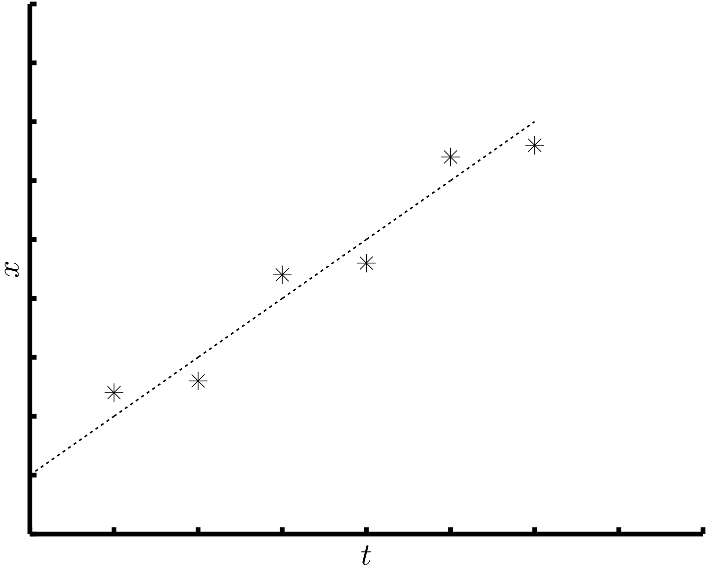

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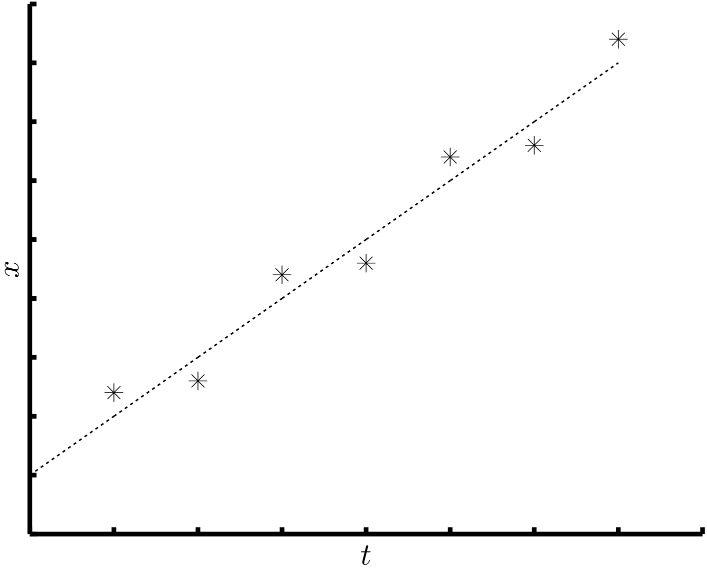

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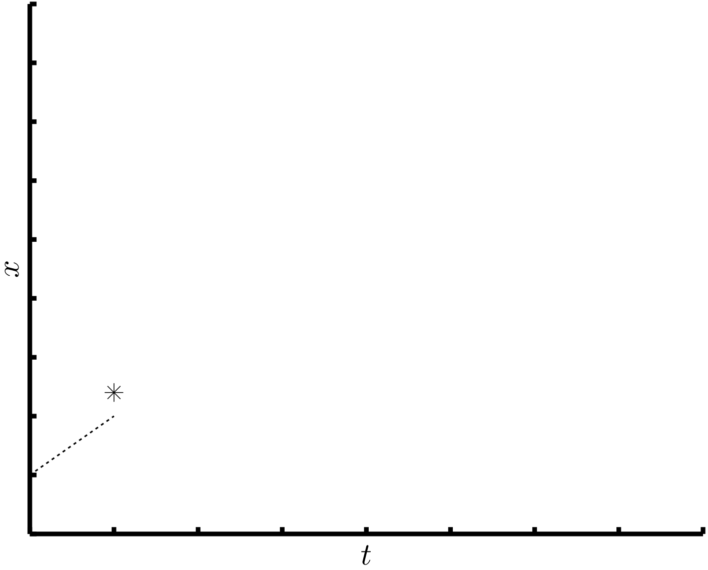

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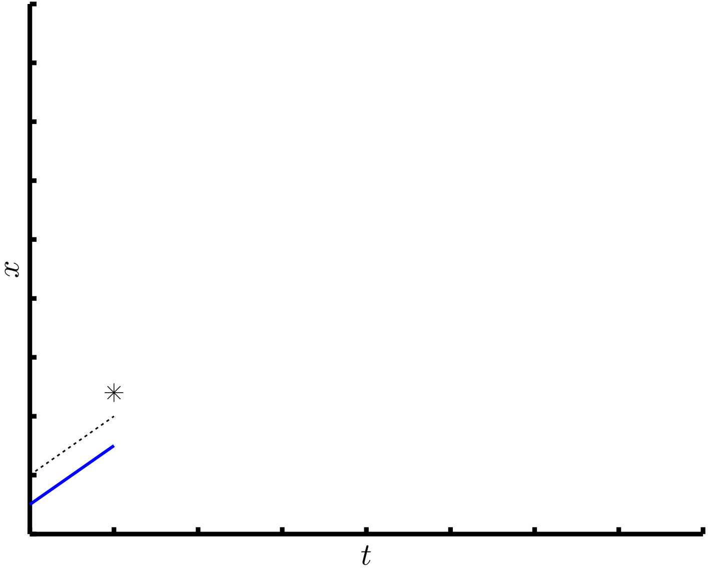

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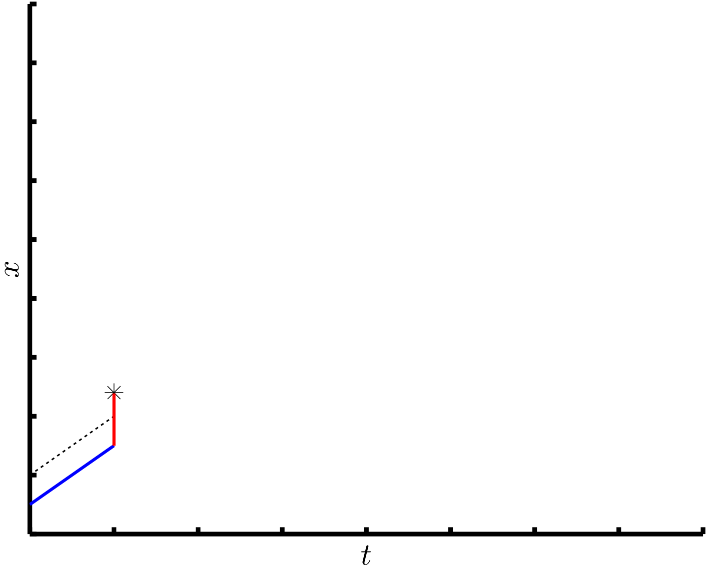

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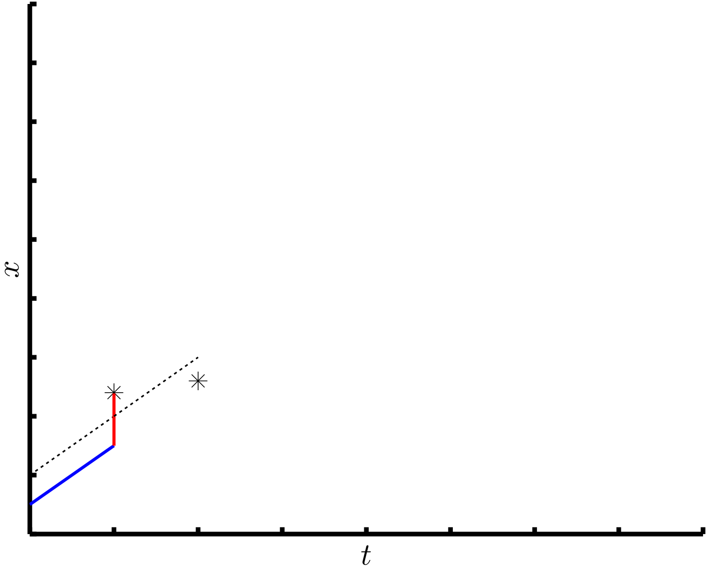

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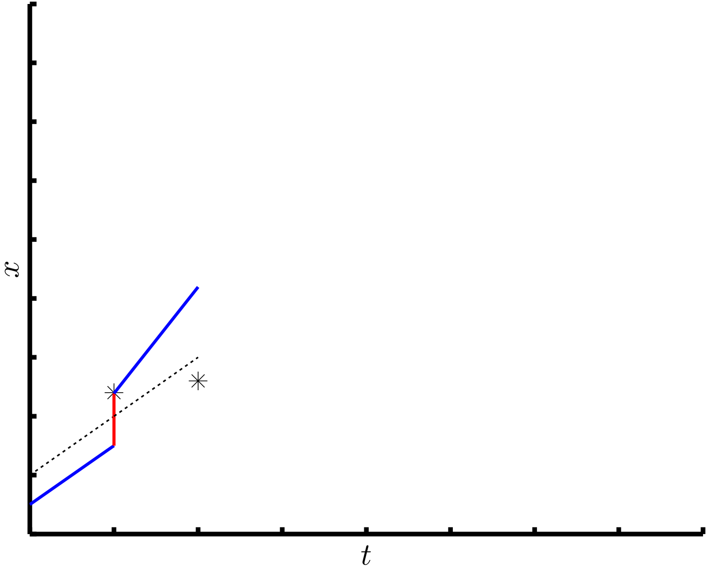

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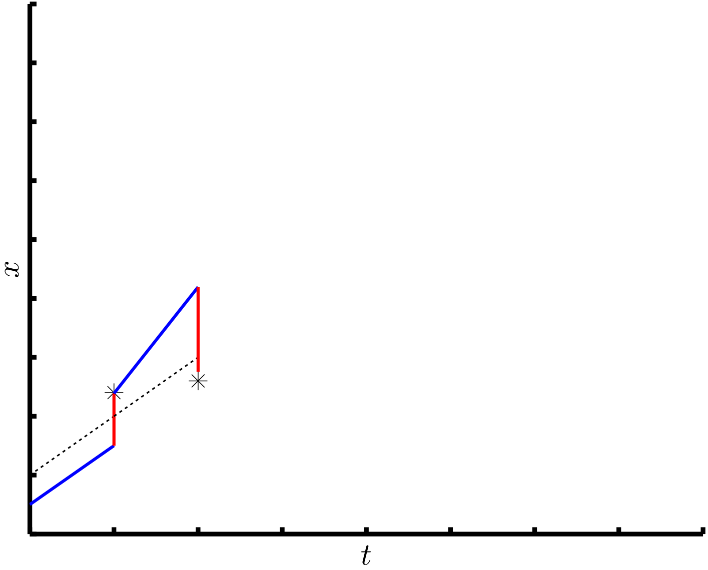

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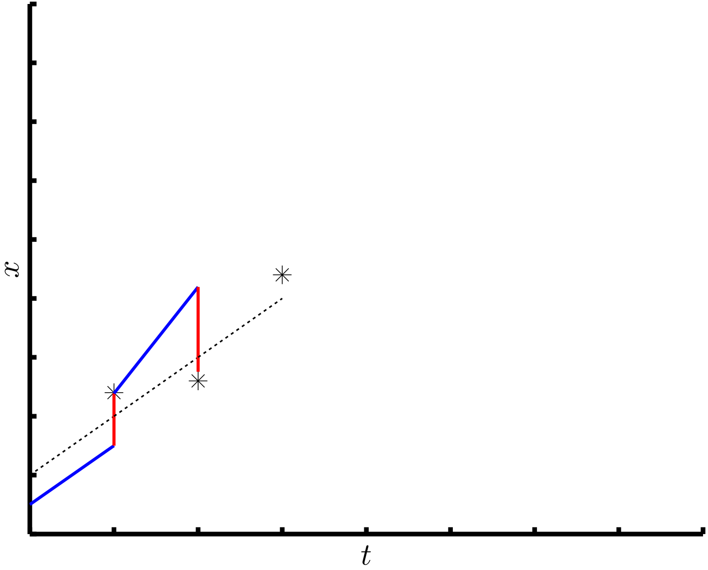

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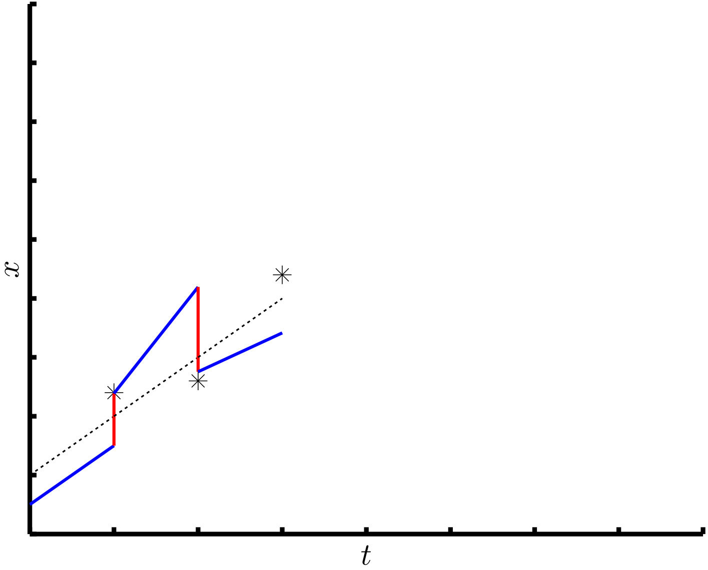

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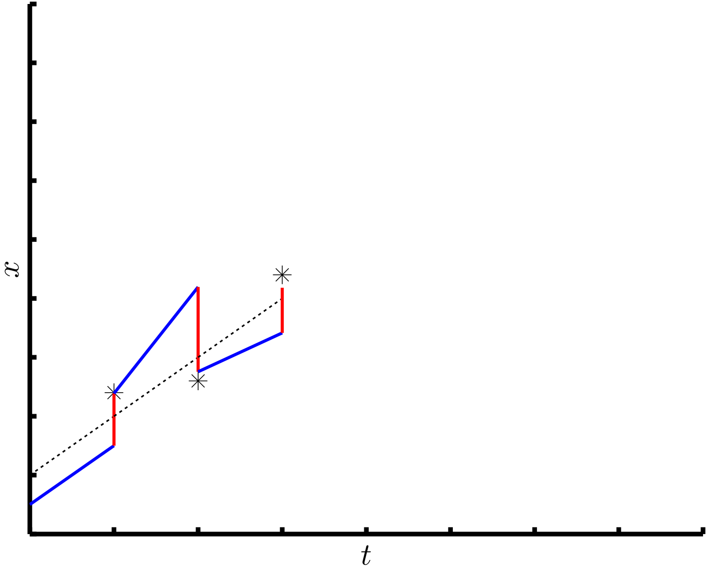

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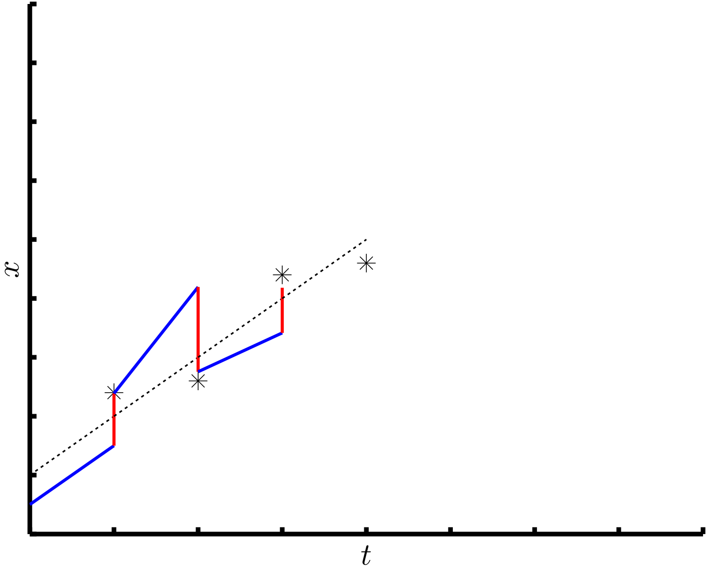

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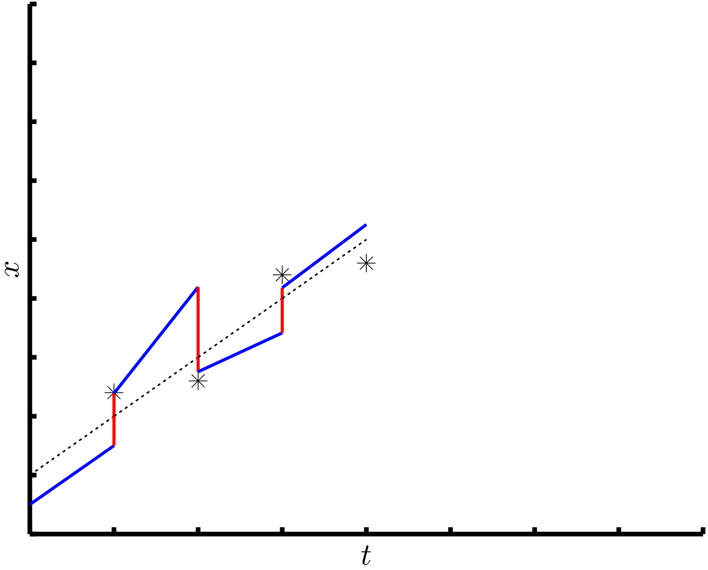

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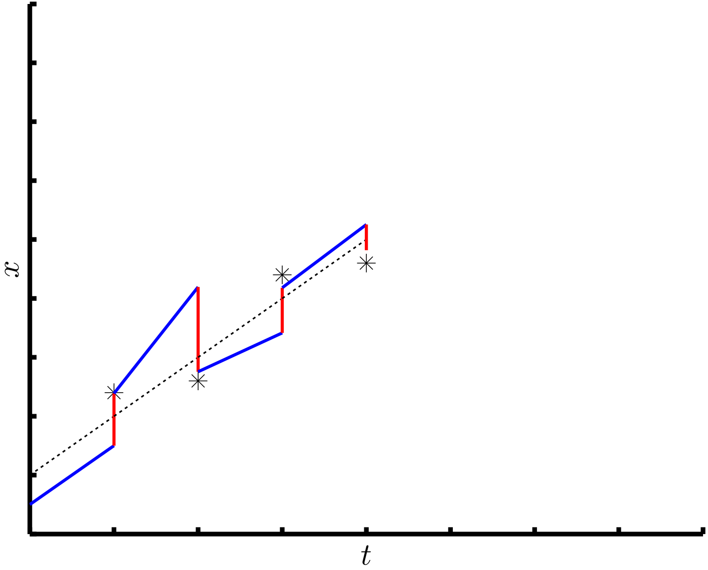

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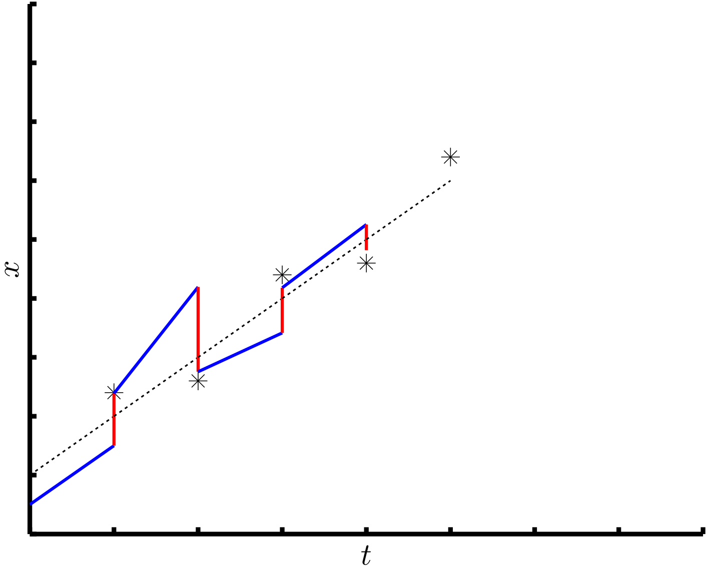

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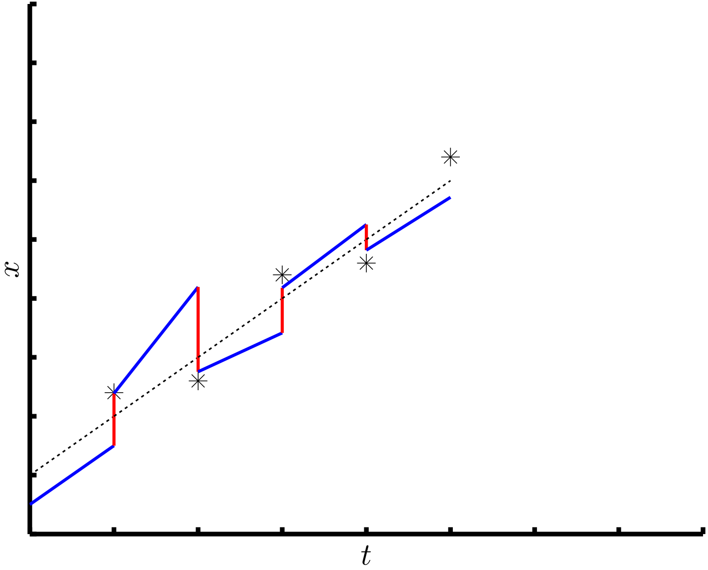

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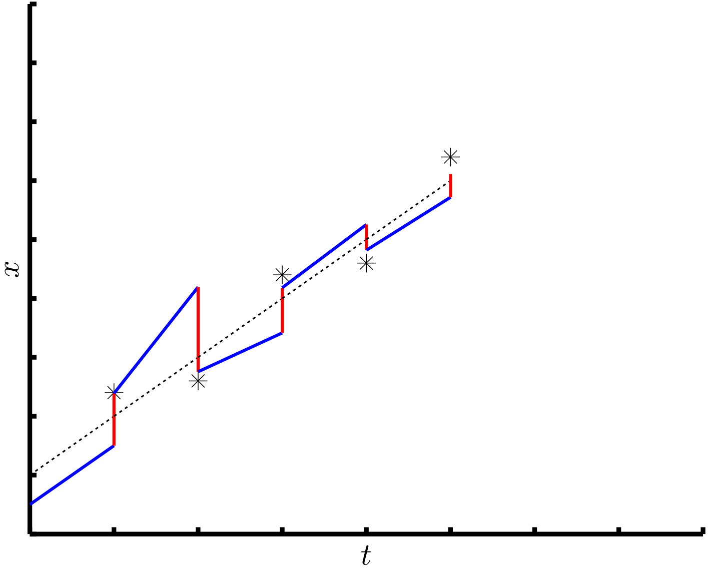

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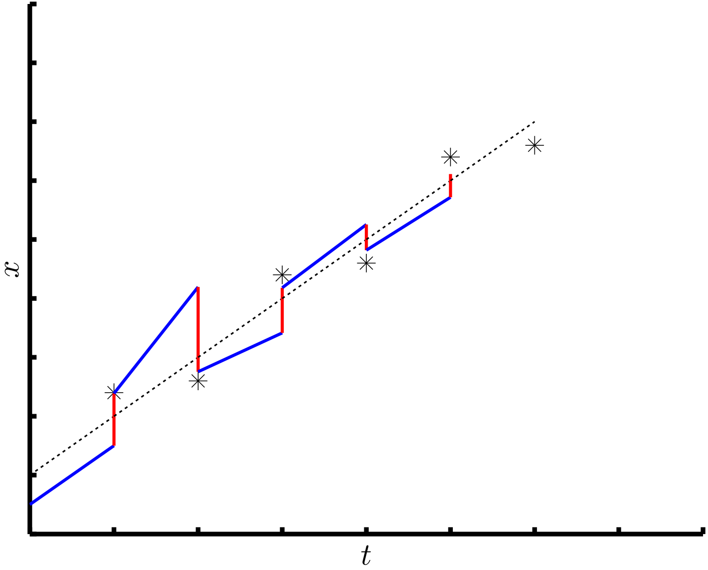

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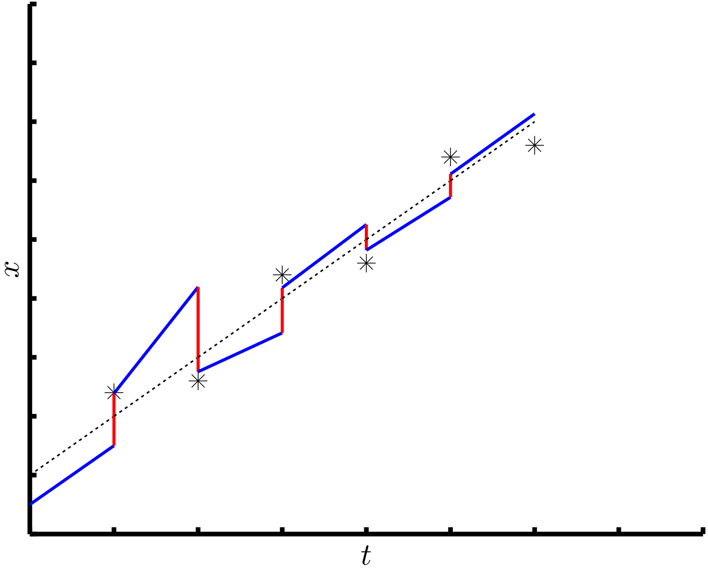

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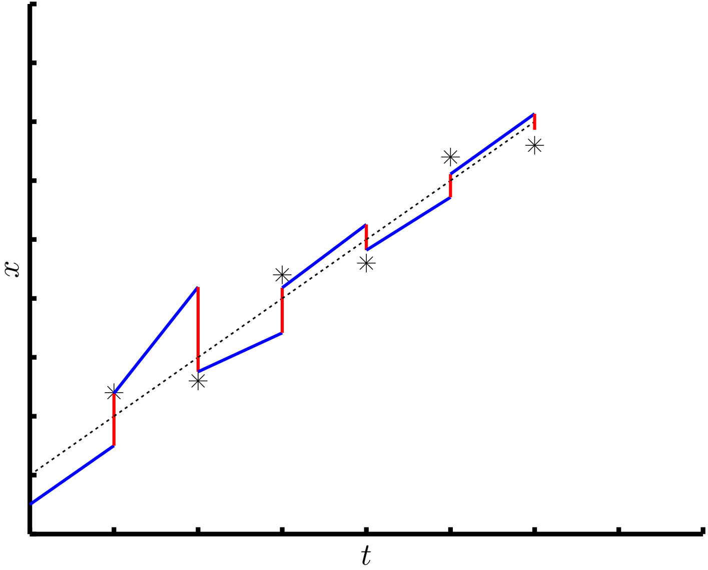

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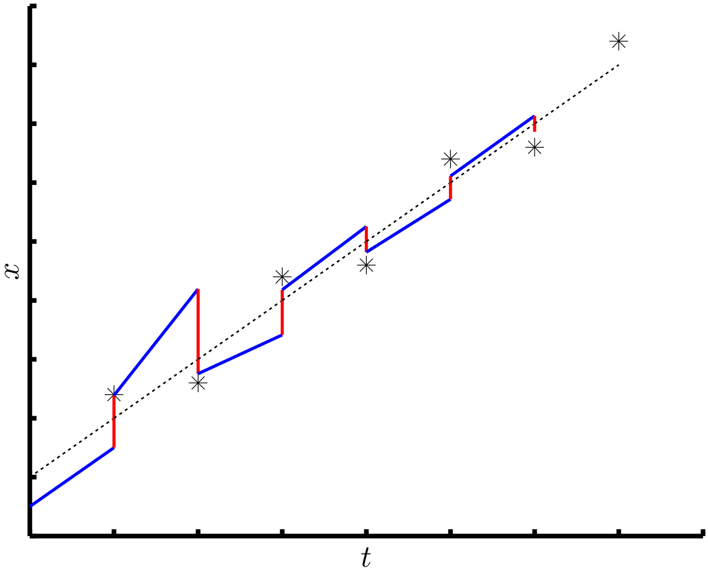

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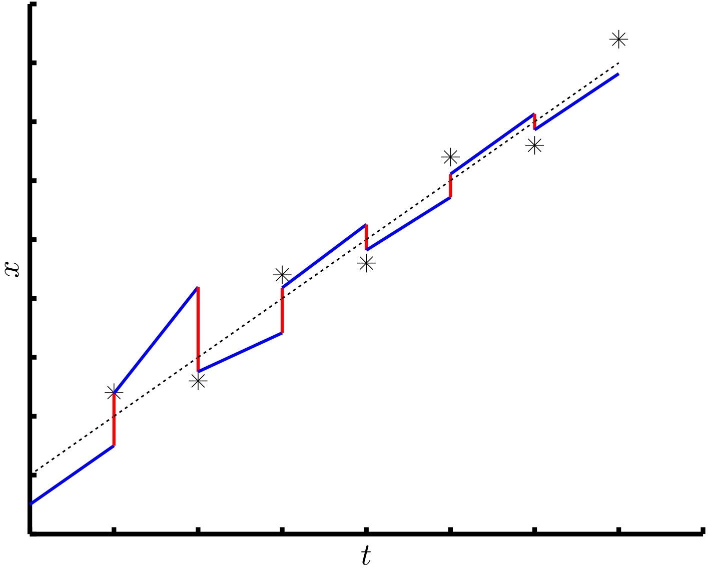

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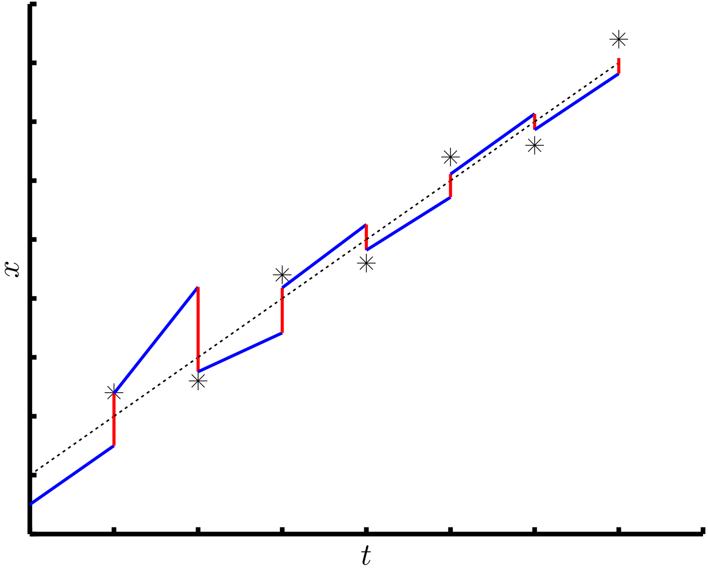
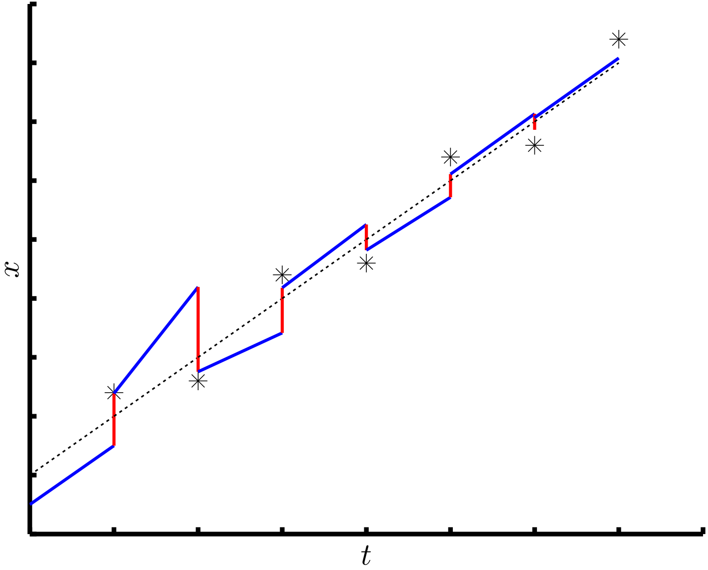


Smoothing

- Reanalysis
- Iterative filtering

EnRTS



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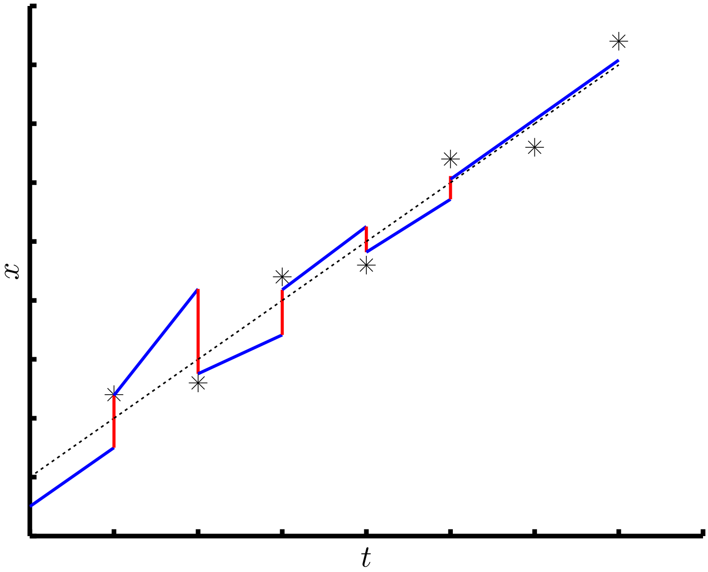

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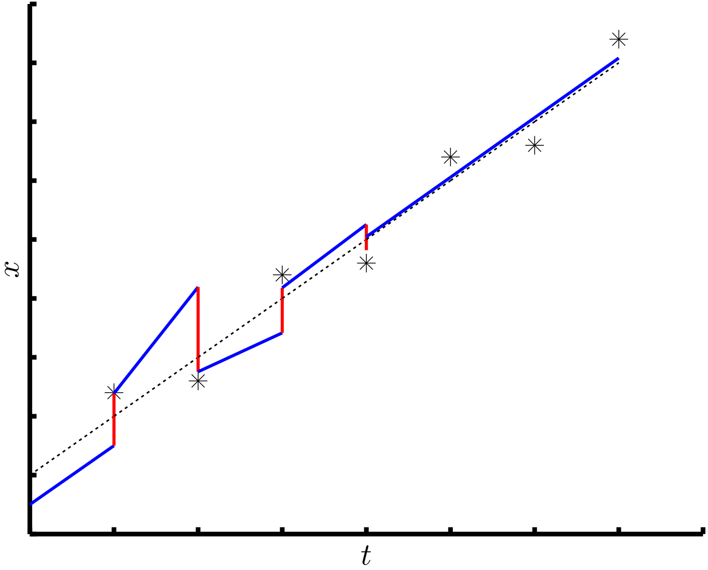

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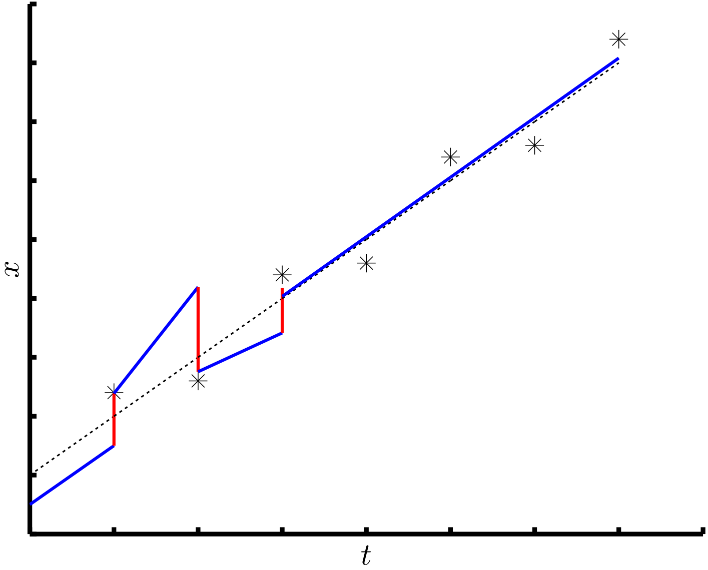

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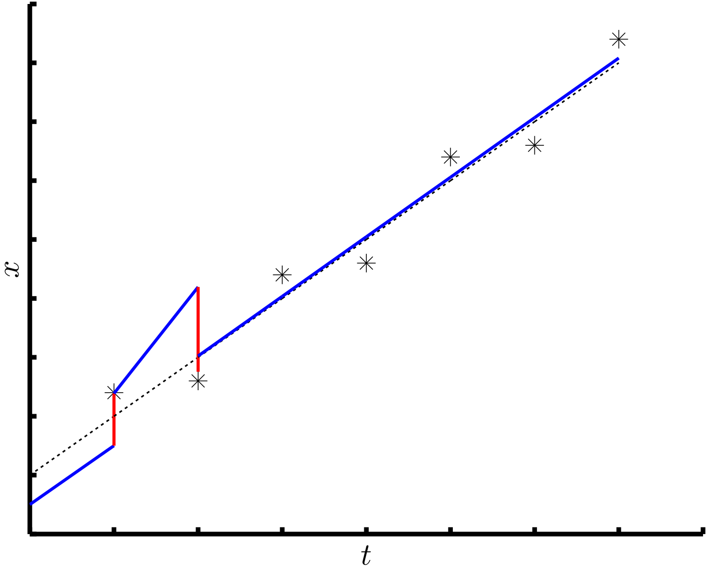

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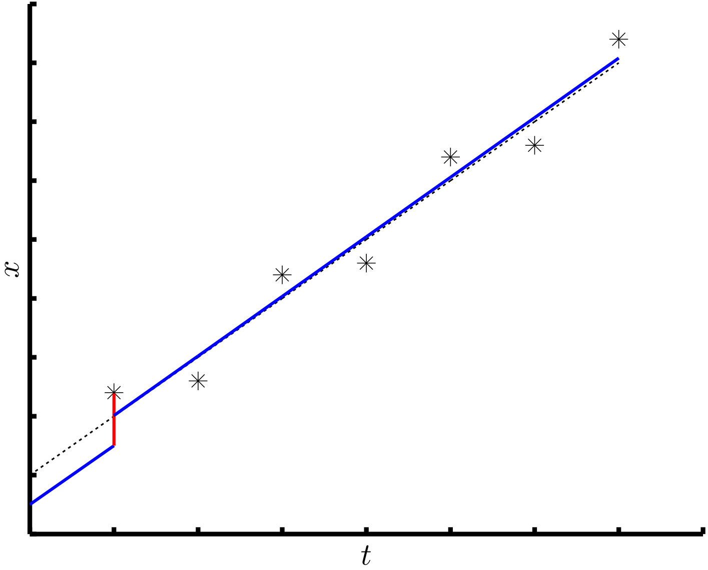

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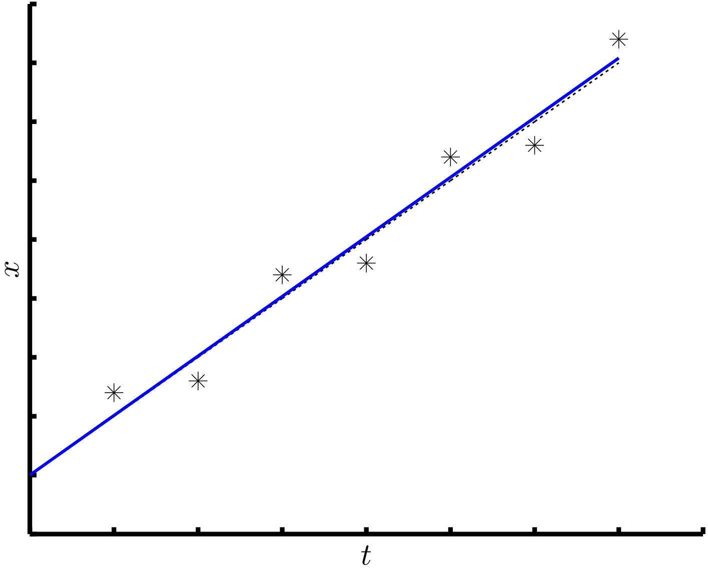
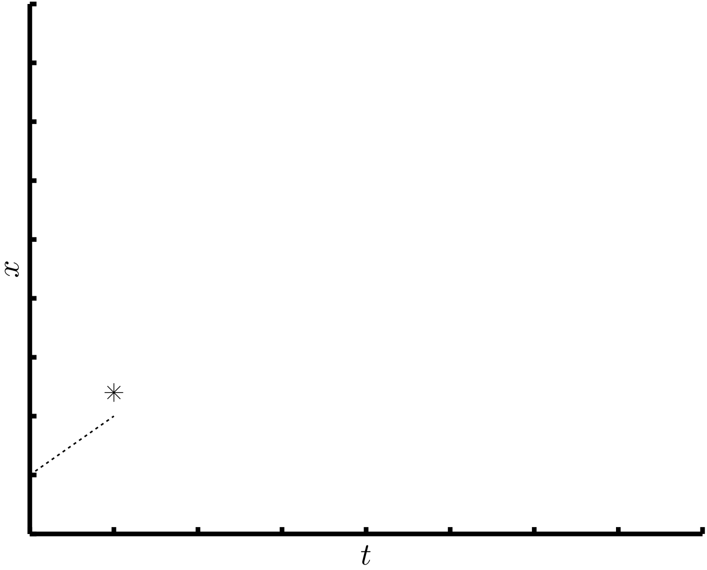


EnKS



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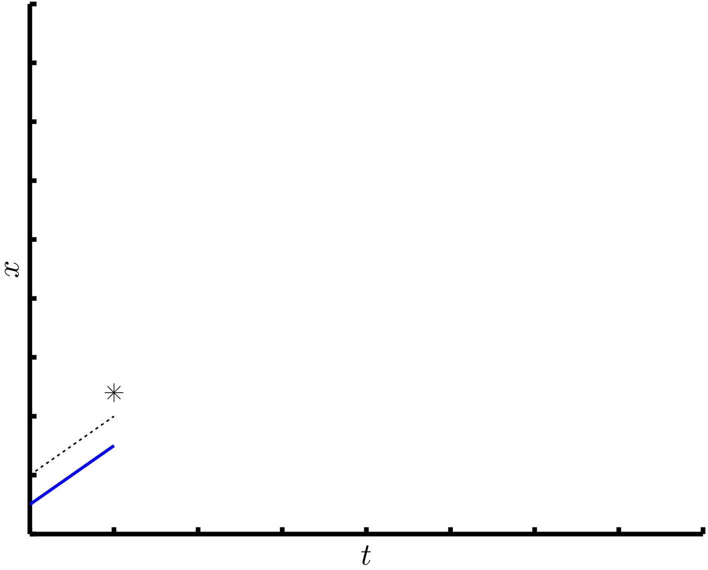

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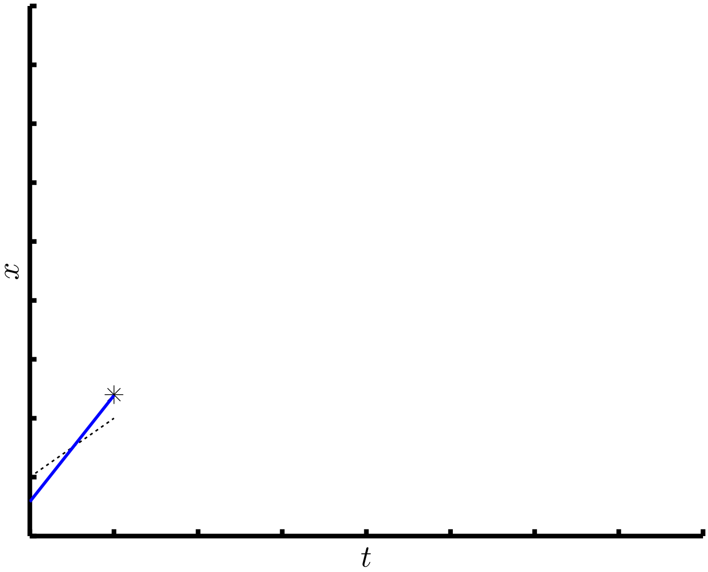

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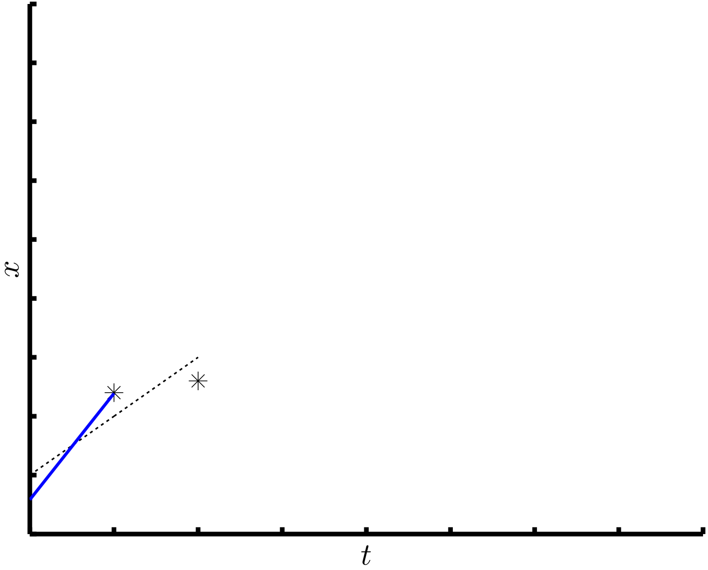

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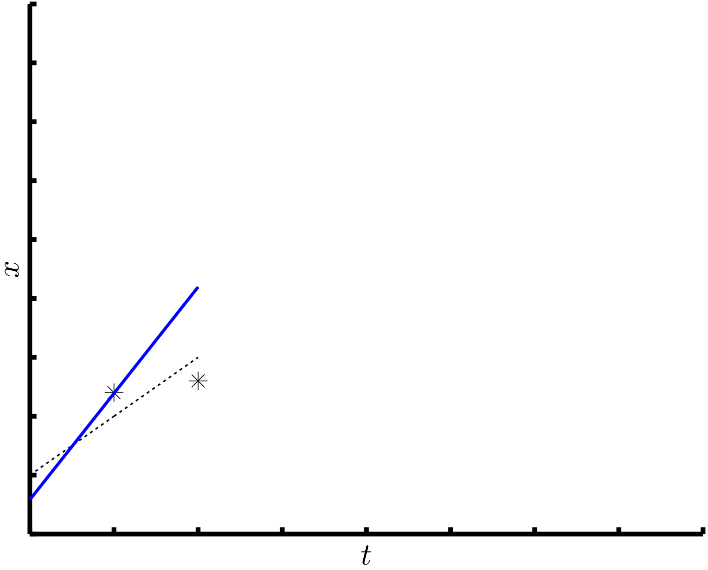

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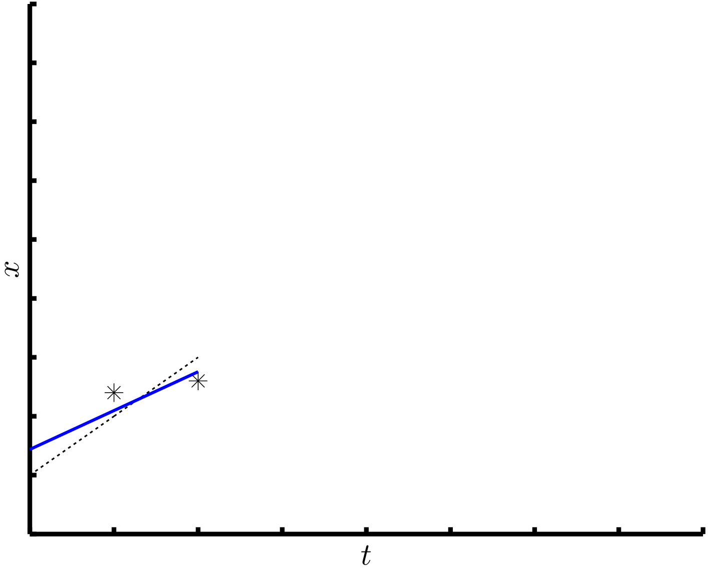

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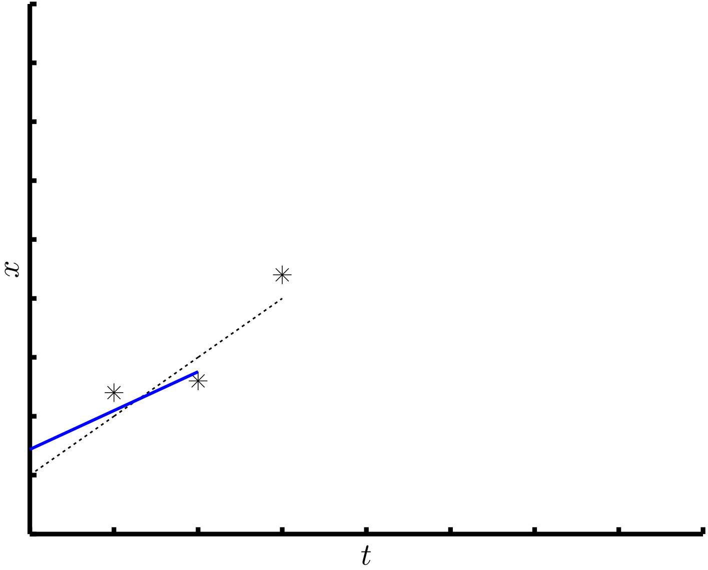

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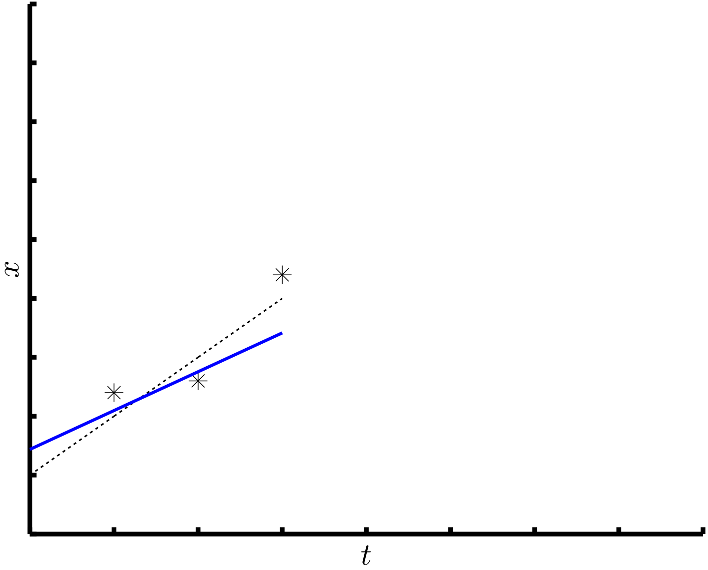

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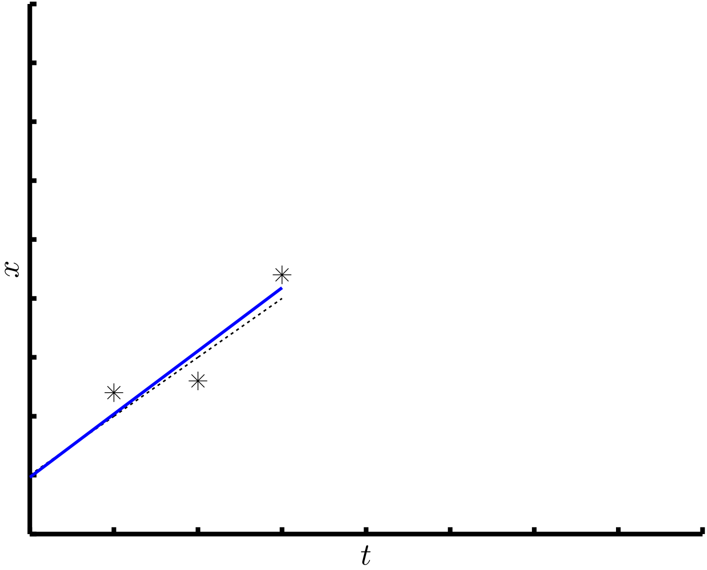

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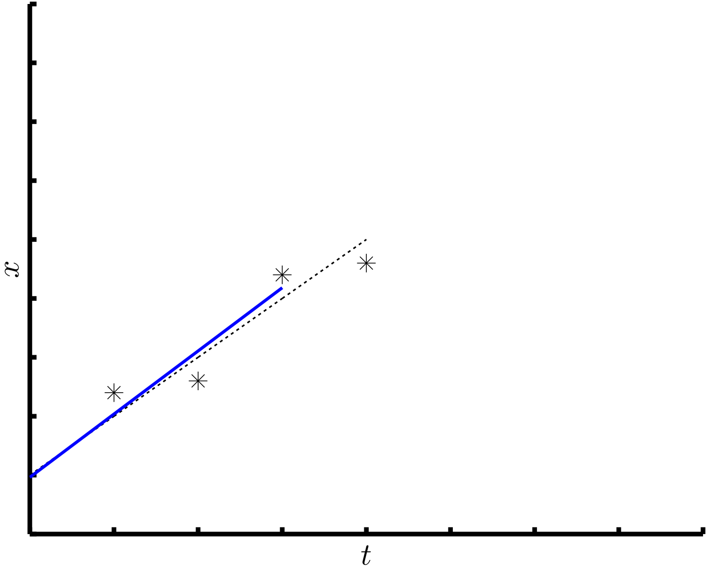

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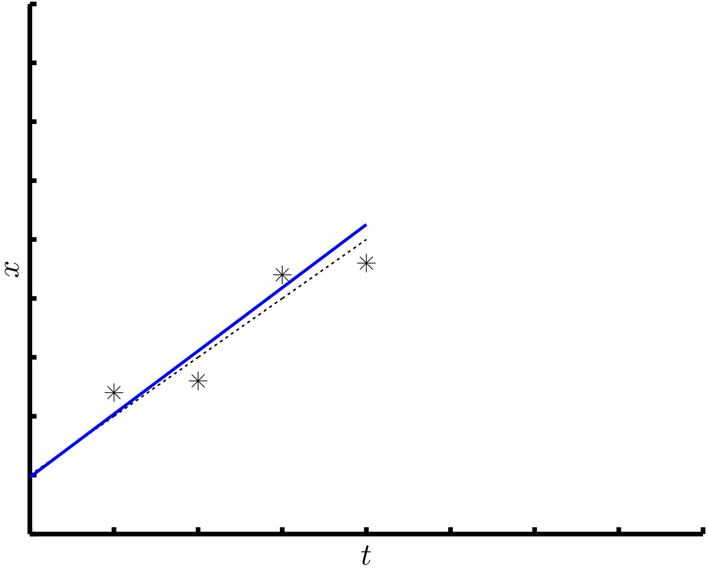

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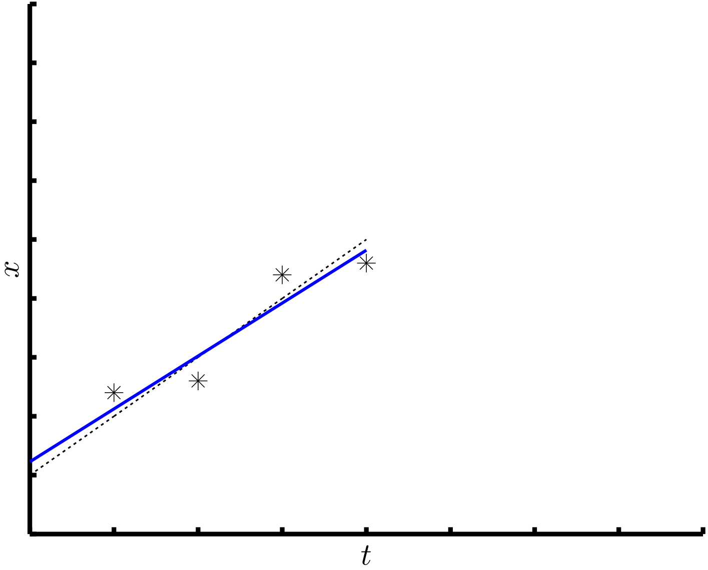

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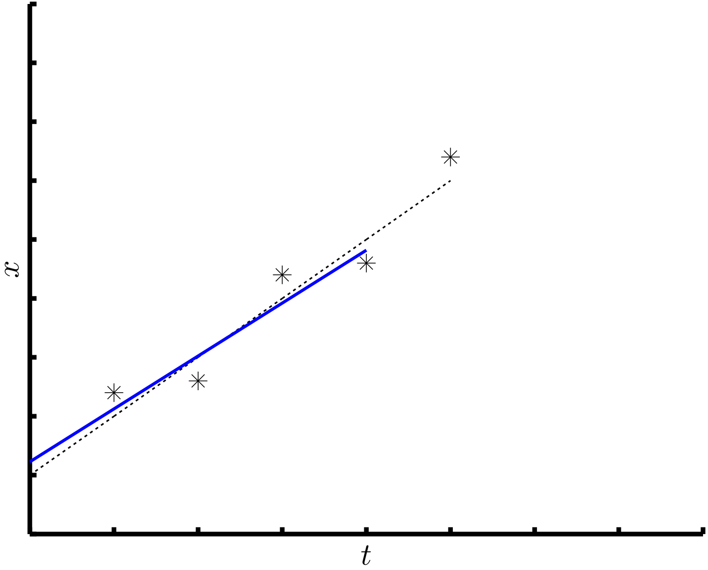

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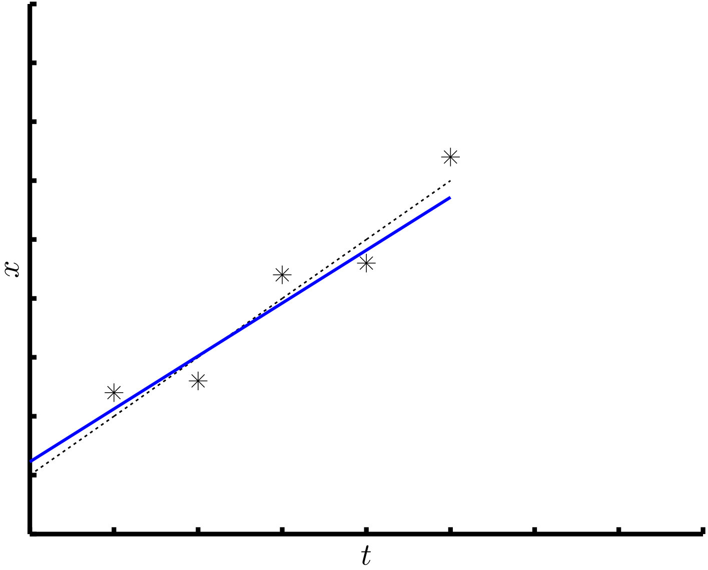

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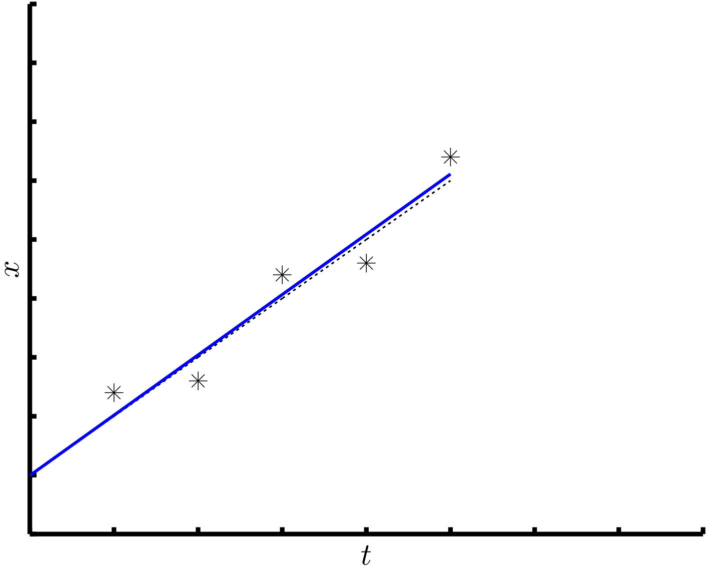

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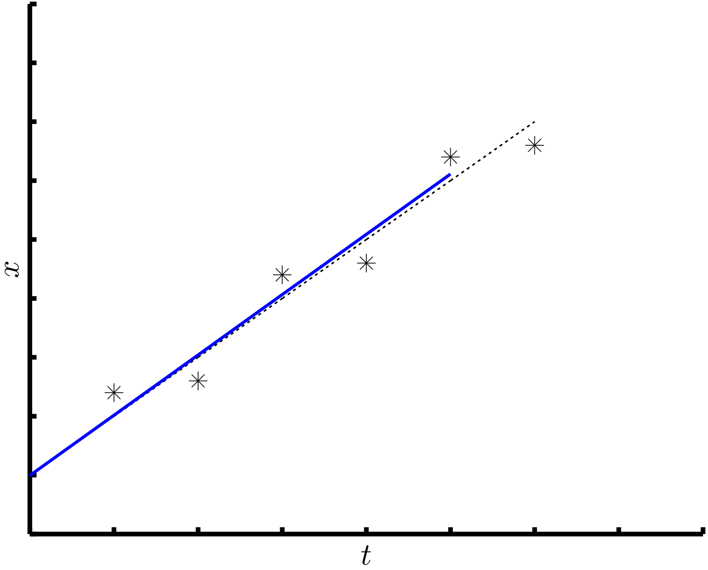

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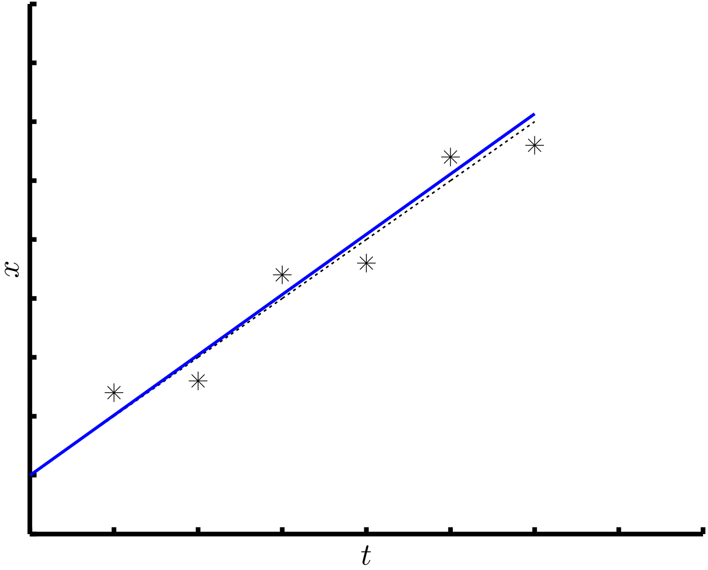

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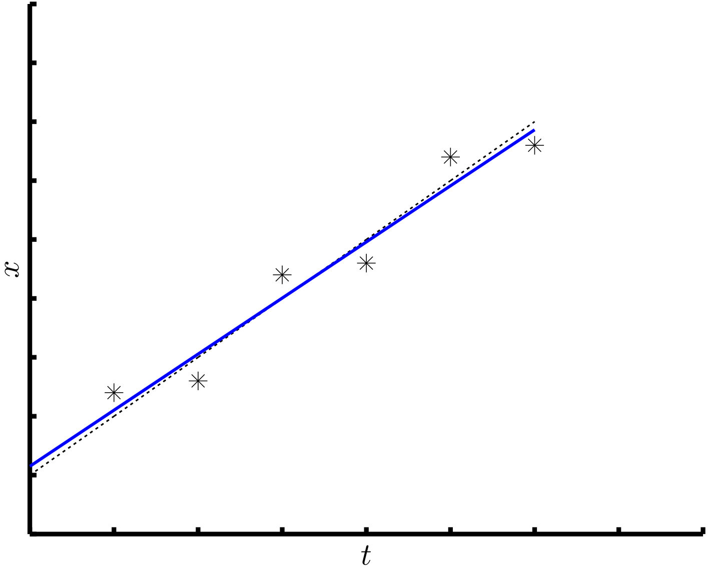

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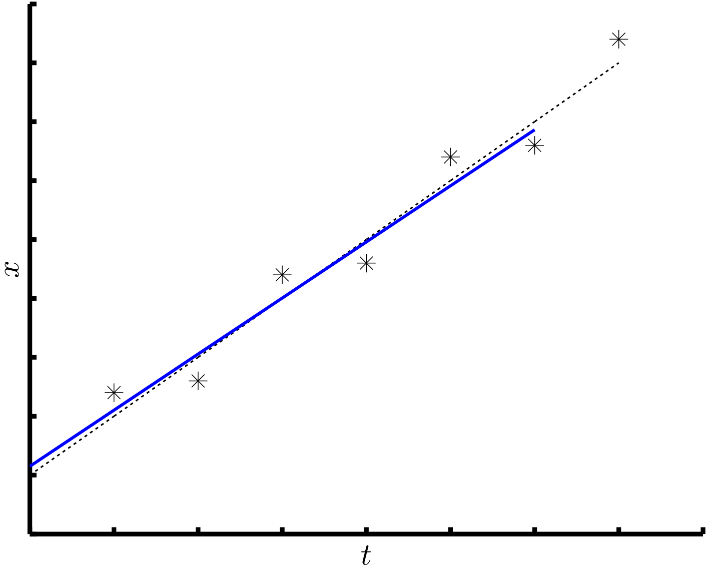

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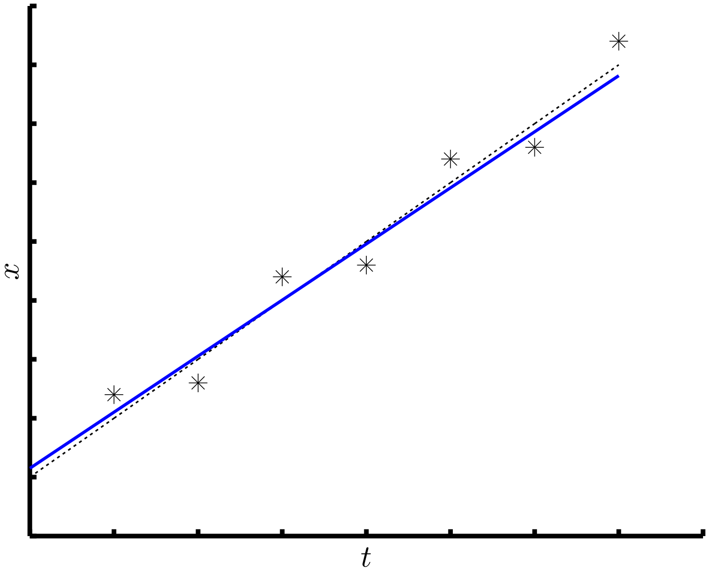

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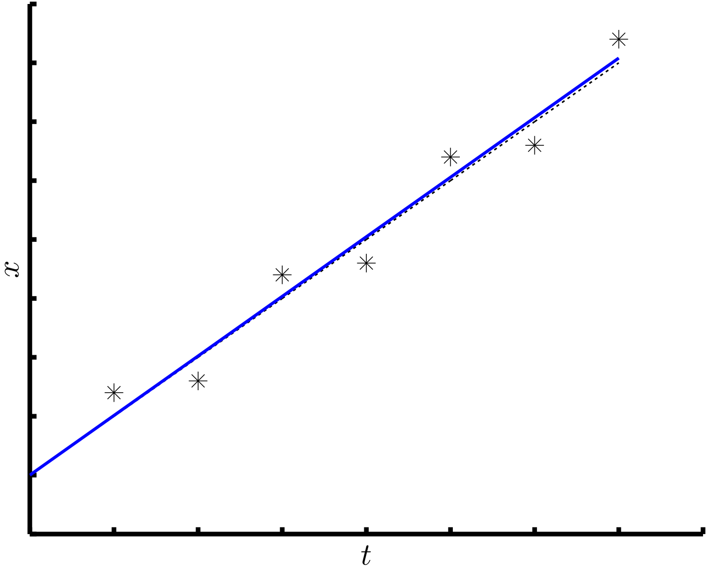
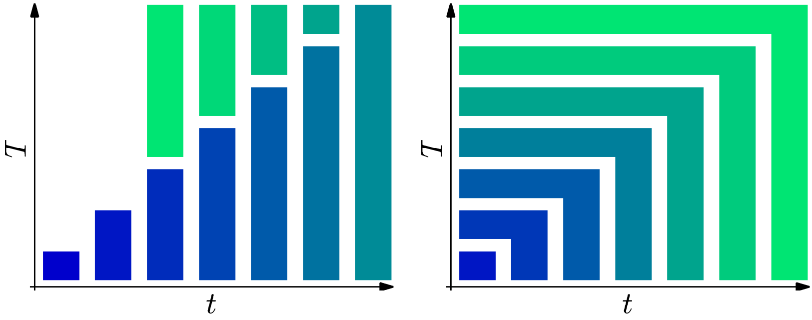


Visualization



EnRTS EnKS

Algorithms

EnRTS

With $\bar{\mathbf{J}}_t = \mathbf{A}_{t|t}\mathbf{A}^+_{t+1|t}$,

$$\mathbf{E}_{t|T} = \mathbf{E}_{t|t} + \bar{\mathbf{J}}_t \left[\mathbf{E}_{t+1|T} - \mathbf{E}_{t+1|t}\right] ,$$

for decreasing t.

EnKS

With $\mathbf{X}^5$ from Evensen'2003

$$\mathbf{E}^{\mathsf{KS}}_{t|T} = \mathbf{E}^{\mathsf{KS}}_{t|T-1}\mathbf{X}^5_T .$$

for increasing T.

Algorithms

EnRTS

With $\bar{\mathbf{J}}_t = \mathbf{A}_{t|t}\mathbf{A}^{+}_{t+1|t}$,

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EnKS

With $\mathbf{X}^5$ from Evensen'2003

$$\mathbf{E}^{\mathsf{KS}}_{t|T} = \mathbf{E}^{\mathsf{KS}}_{t|T-1}\mathbf{X}^5_T .$$

for increasing T.

Proof

Lemma: The EnRTS on-line

Unconditionally,

$$\mathbf{E}_{t|T} = \mathbf{E}_{t|t} + \sum_{k=t+1}^{T} \left(\prod_{\tau=t}^{k-1} \bar{\mathbf{J}}_\tau \right) \left[\mathbf{E}_{k|k} - \mathbf{E}_{k|k-1} \right]$$

Lemma: $\bar{\mathbf{J}}_\tau$ recursively

Providing $N \leq m$ (or linear dynamics),

$$\prod_{\tau=t}^{T-1} \bar{\mathbf{J}}_\tau = \mathbf{A}_{t|T-1} \mathbf{A}_{T|T-1}^{+} \,.$$

Theorem: Equivalence

$$\mathbf{E}_{t|T} = \mathbf{E}_{t|T}^{\mathsf{KS}} \,.$$

Proof

Lemma: The EnRTS on-line

Unconditionally,

$$\mathbf{E}_{t|T} = \mathbf{E}_{t|t} + \sum_{k=t+1}^{T} \left(\prod_{\tau=t}^{k-1} \bar{\mathbf{J}}_\tau \right) \left[\mathbf{E}_{k|k} - \mathbf{E}_{k|k-1} \right]$$

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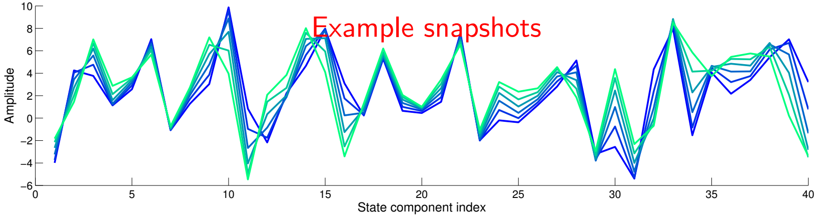
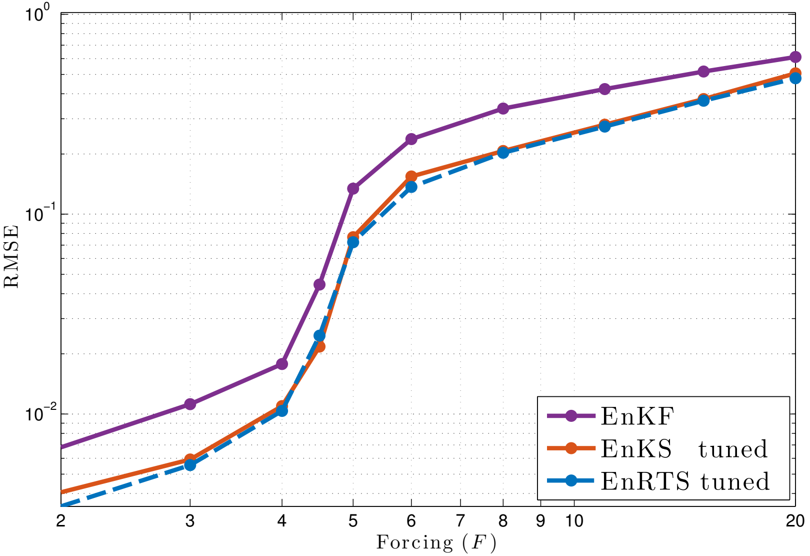
Lorenz-96 system

Integrated with RK4: $$\frac{dx_i}{dt} = (x_{i+1} - x_{i-2})\, x_{i-1} - x_i + F$$



$$\text{RMSE} = \frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{1}{m}\|\bar{\boldsymbol{x}}_t - \boldsymbol{x}_t\|_2^2}\,.$$

In practice



10^0
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RMSE
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Forcing (F)
EnKF
EnKS tuned
EnRTS tuned

Summary

- Both smoothers can be formulated on- and off- line
- If $N < m$: equivalence
- Equivalence broken by ad-hoc tuning
- But capability remains equal

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Paper

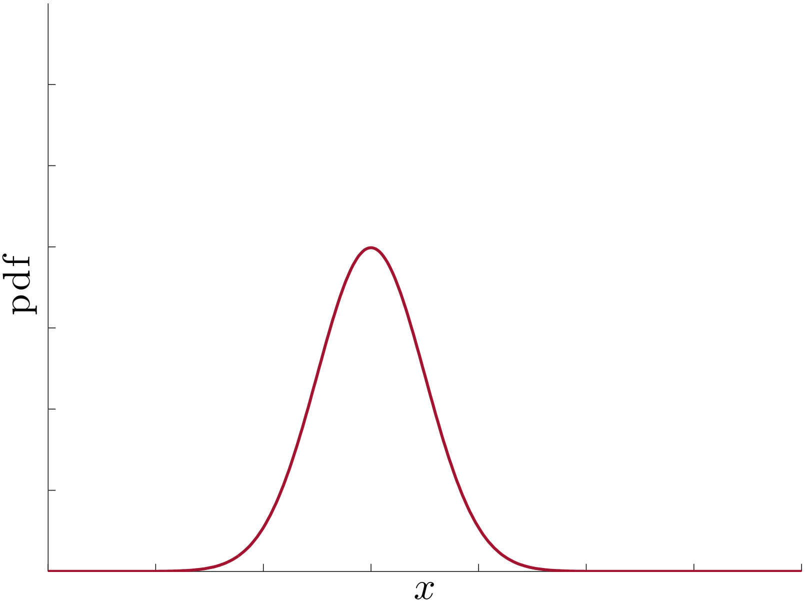
Patrick Nima Raanes.
On the ensemble Rauch-Tung-Striebel smoother and its equivalence to the ensemble Kalman smoother.
Quarterly Journal of the Royal Meteorological Society, **2015**.

The Rauch-Tung-Striebel (RTS) smoother is a linear-Gaussian smoothing algorithm that is popular in the engineering community. This note is a study of its ensemble formulation (EnRTS). An on-line expression is derived and discussed. In particular, it is used to show that the EnRTS is equivalent to the ensemble Kalman smoother (EnKS), even in the nonlinear, non-Gaussian case. The theory is revisited under practical considerations and equability is illustrated by numerical experiments, even though equivalence is broken by inflation and localisation.

The EnKF-N and inflation (poster teaser)

Patrick Nima Raanes
Marc Bocquet

11th EnKF workshop, Ulvik, June 19, 2016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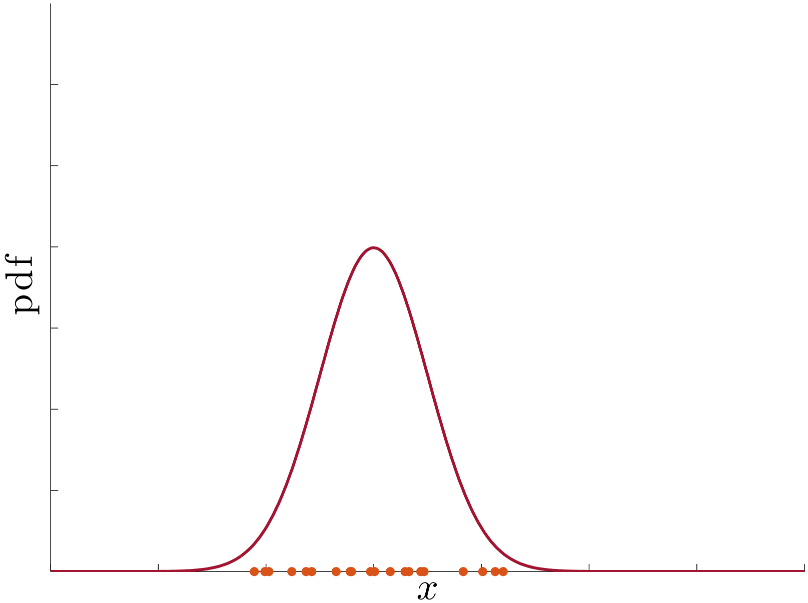

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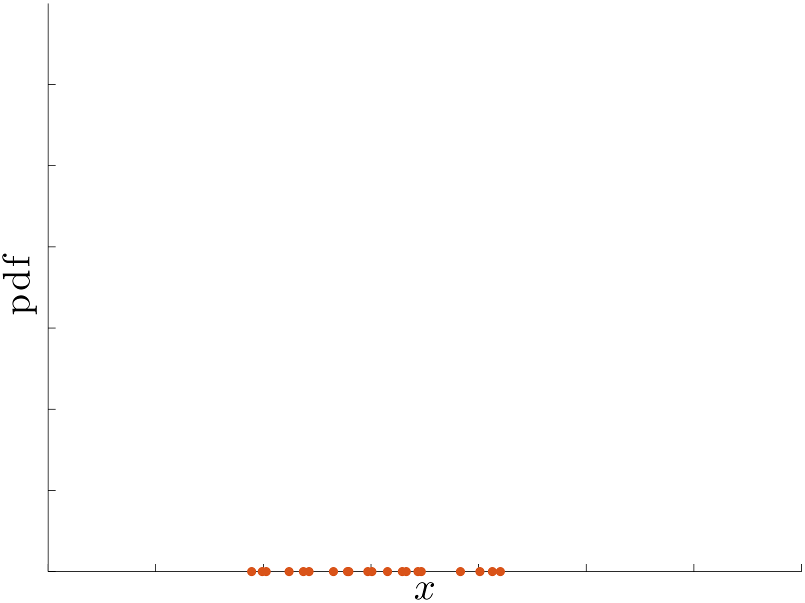
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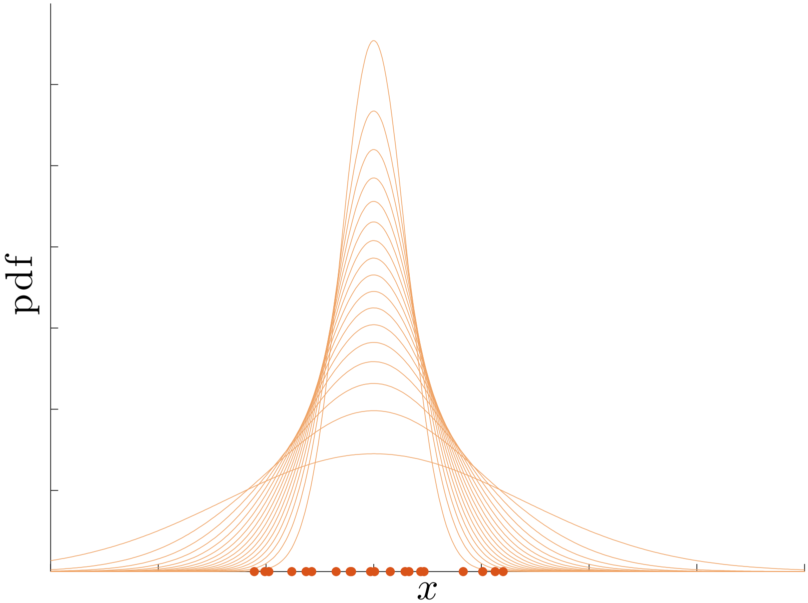

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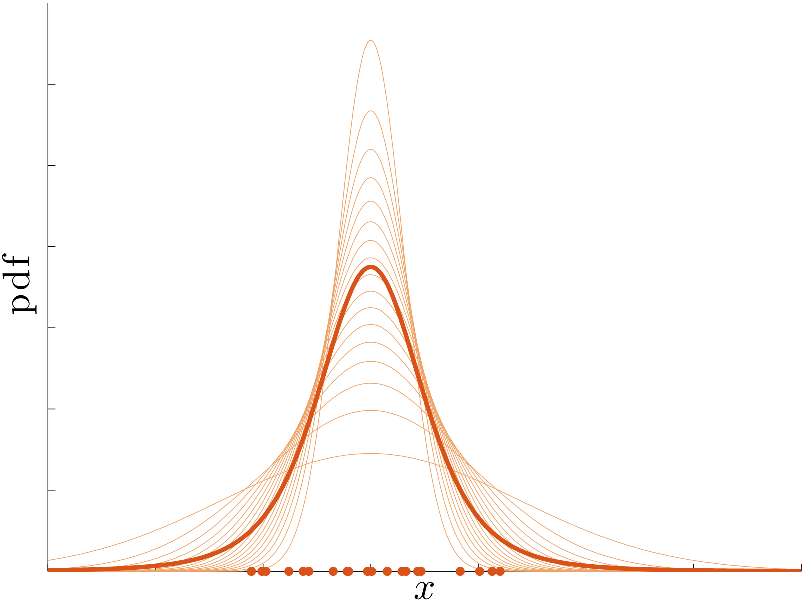

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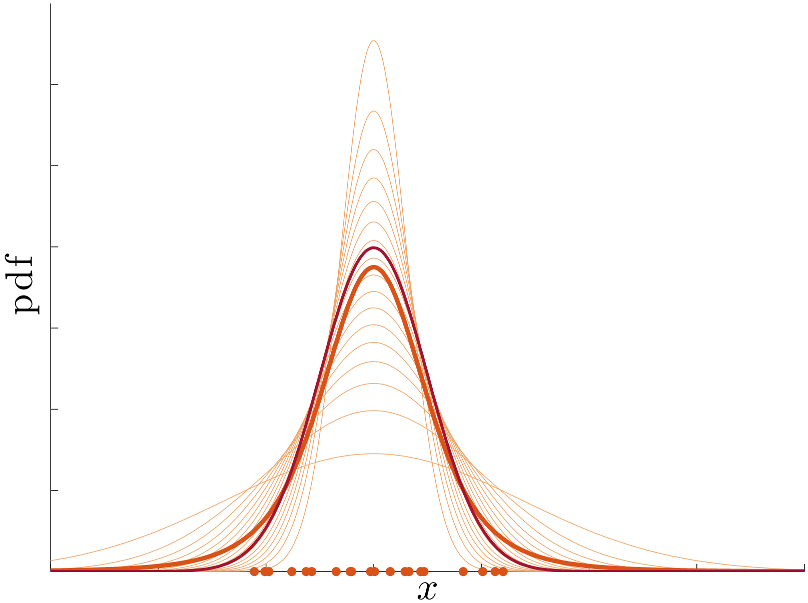

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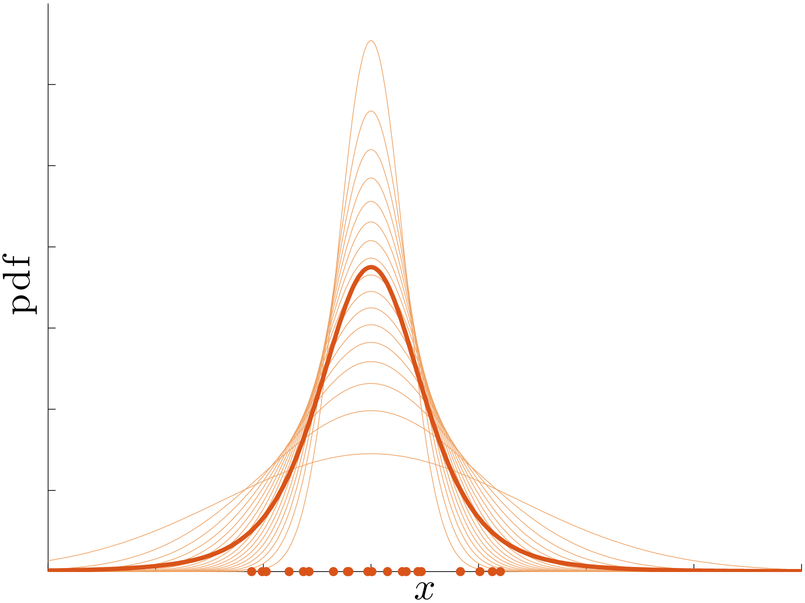

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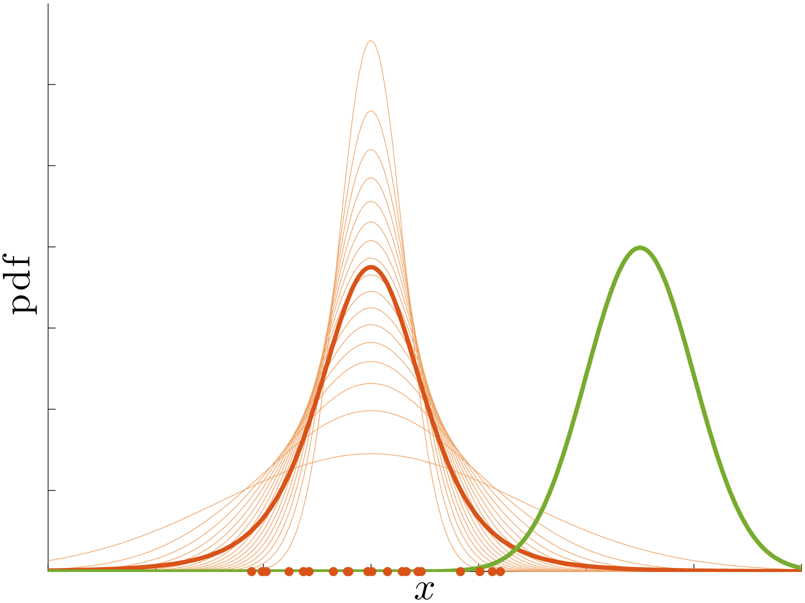
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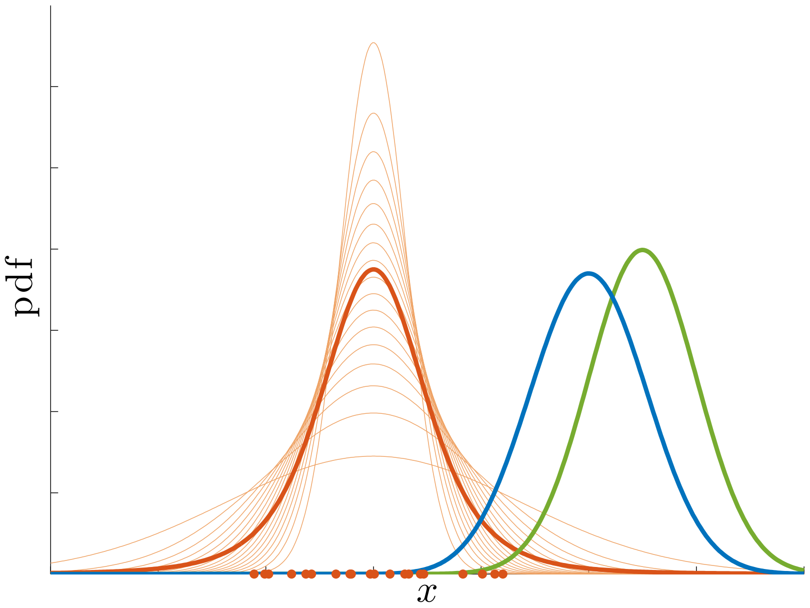
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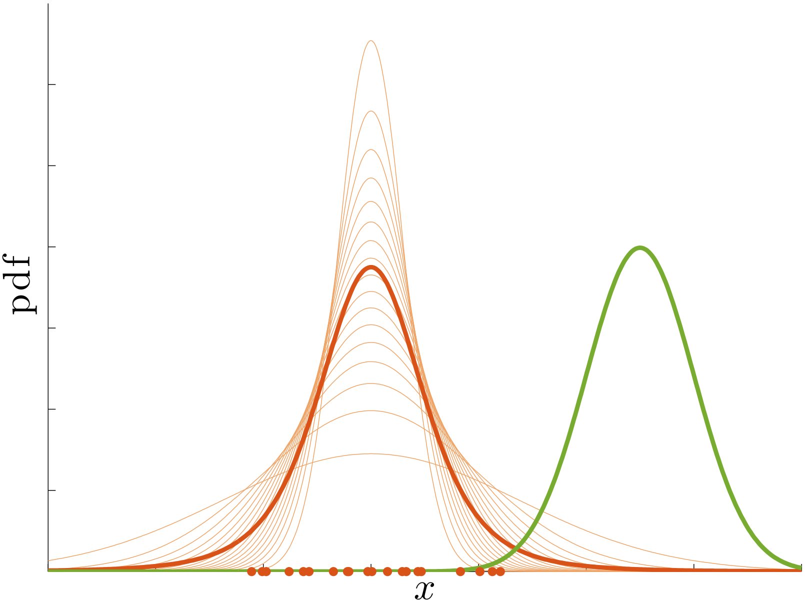
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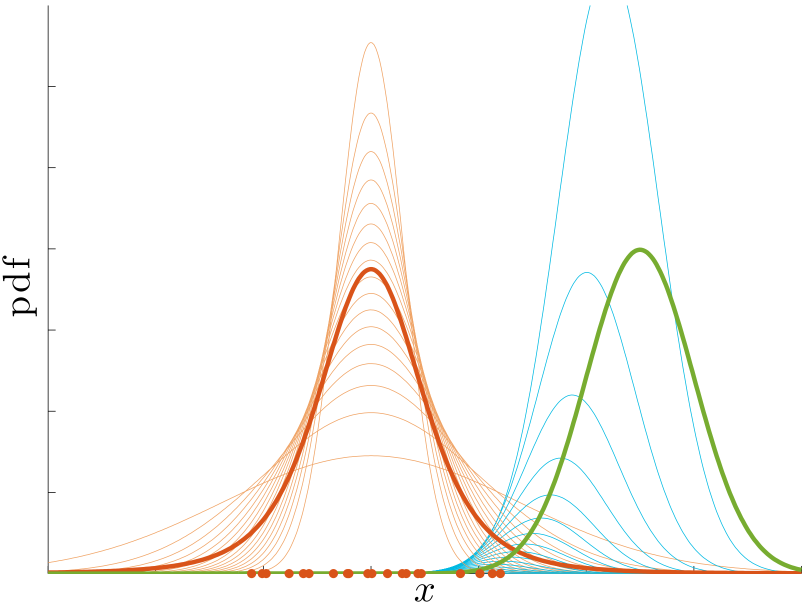
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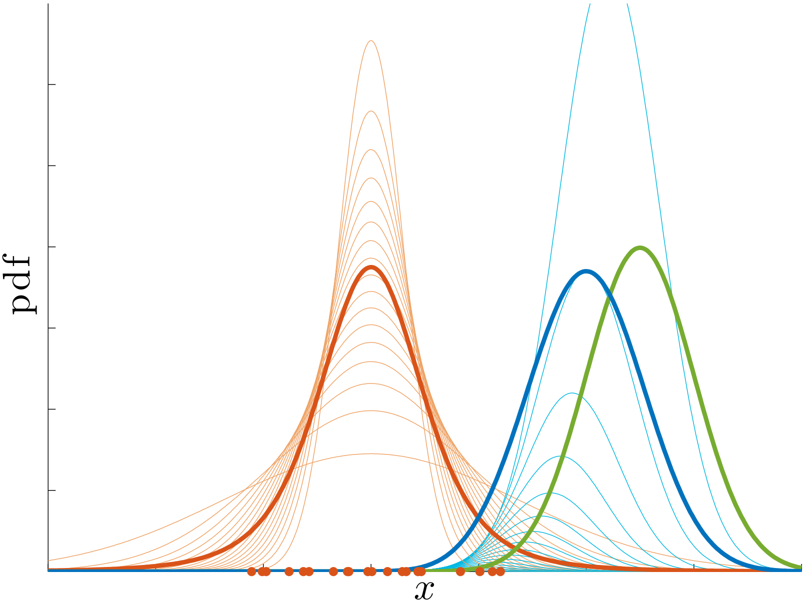
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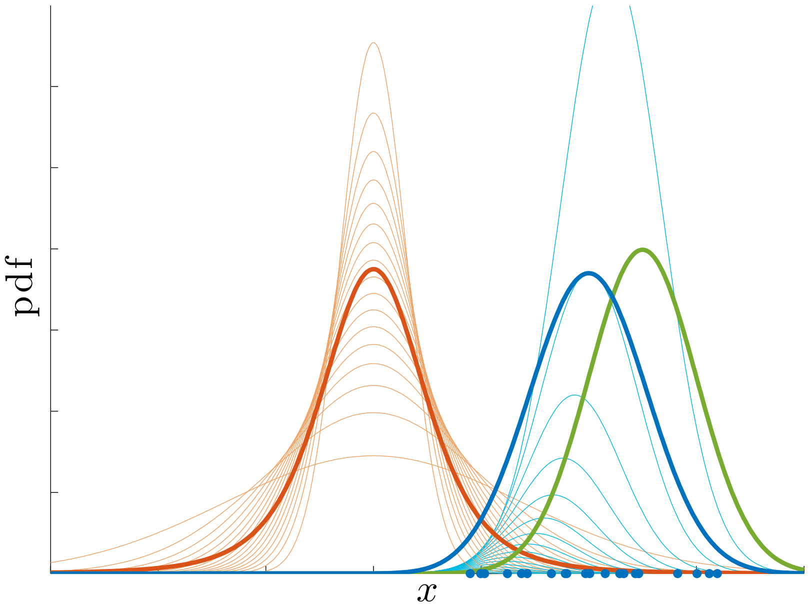

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Summary

- Less dogmatic assumptions $\Longrightarrow$ EnKF-N
  - Posterior variance depends on innovation
  - Careful about parameterization and implicit assumptions
- Dual perspective: scale mixture
  - Adaptive inflation
- Primal perspective: Student t prior

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    - Better than "unbiased"
    - Sequential feedback
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  - Good performance, no additional cost
  - Future: estimate model error inflation
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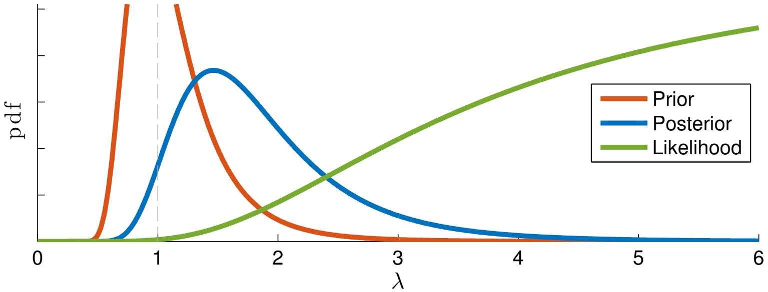
Paper

Marc Bocquet, Patrick N. Raanes, and Alexis Hannart.
Expanding the validity of the ensemble Kalman filter without the intrinsic need for inflation.
Nonlinear Processes in Geophysics, 22(6):645-662, **2015**.

Appendix



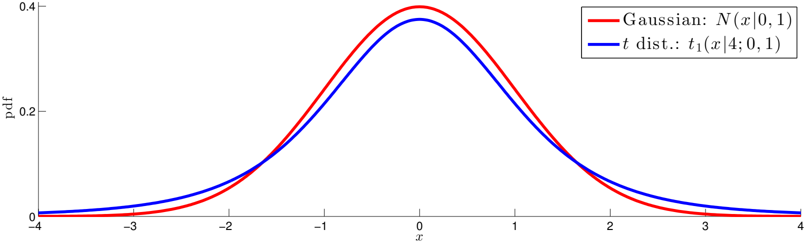
Prior:	$p(\lambda^2 \vert \mathbf{E}) = \chi^{-2}(\lambda^2 \vert N-1)$
Likelihood:	$p(\boldsymbol{y}, \boldsymbol{w}_\star \vert \mathbf{E}, \lambda^2) = \exp\left(-\frac{1}{2}\Vert\bar{\boldsymbol{\delta}}\Vert^2_{\mathbf{Y}\mathbf{Y}^\mathsf{T}/\zeta+\mathbf{R}}\right)$
Posterior:	$p(\boldsymbol{w}_\star, \zeta \vert \mathbf{E}, \boldsymbol{y}) = \exp\left(-\frac{1}{2}\,D(\zeta)\right)$

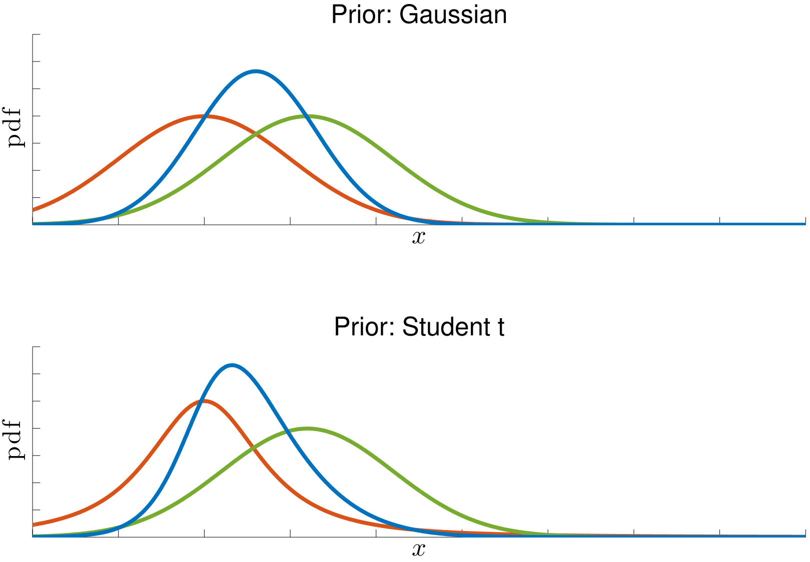
Gaussian distribution

$$\mathcal{N}(x|0,1) \propto e^{-\frac{1}{2}x^2}$$

(Student) t distribution

$$\boldsymbol{t}_1(x|1;0,1) \propto \frac{1}{1+x^2}$$





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Prior: Student t
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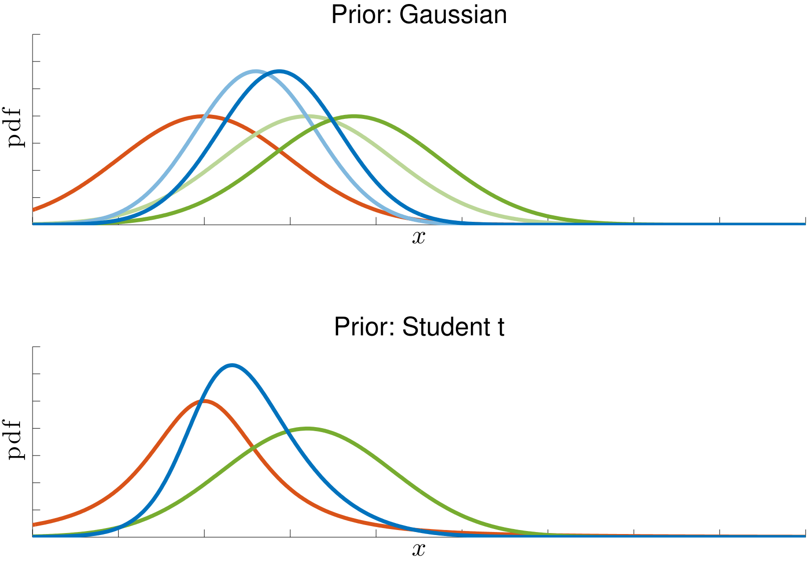

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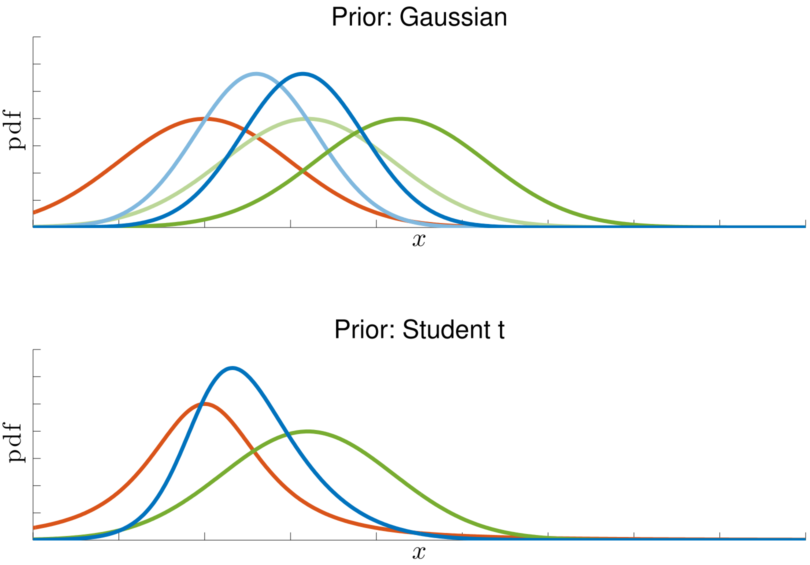
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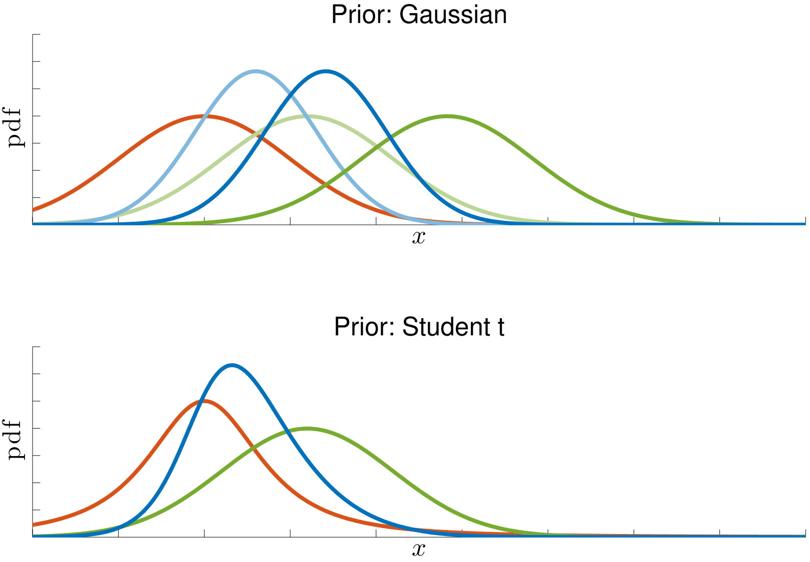

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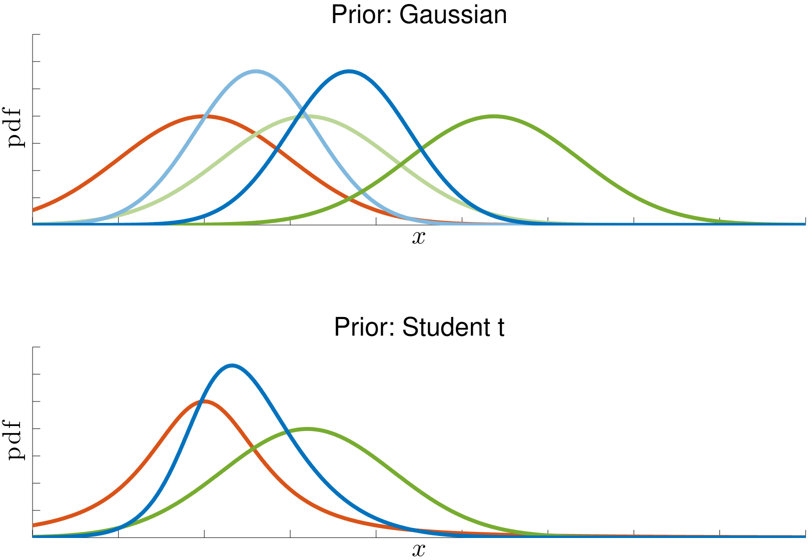
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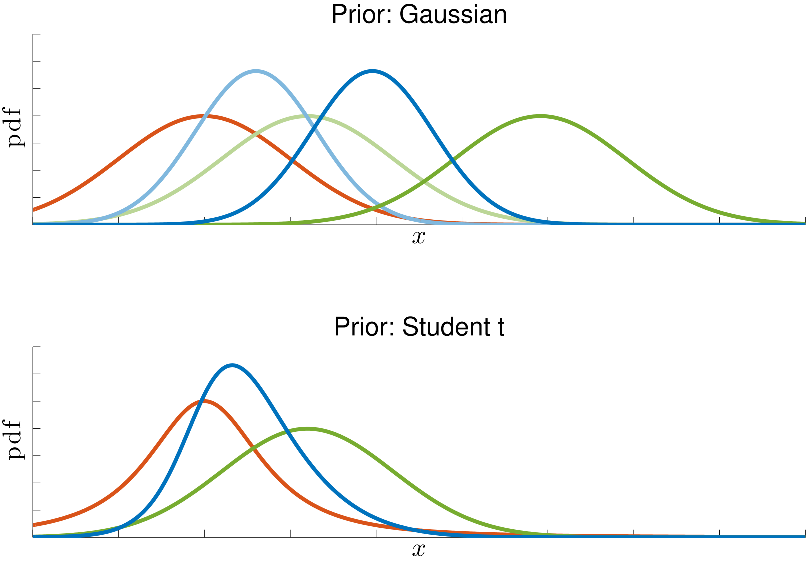
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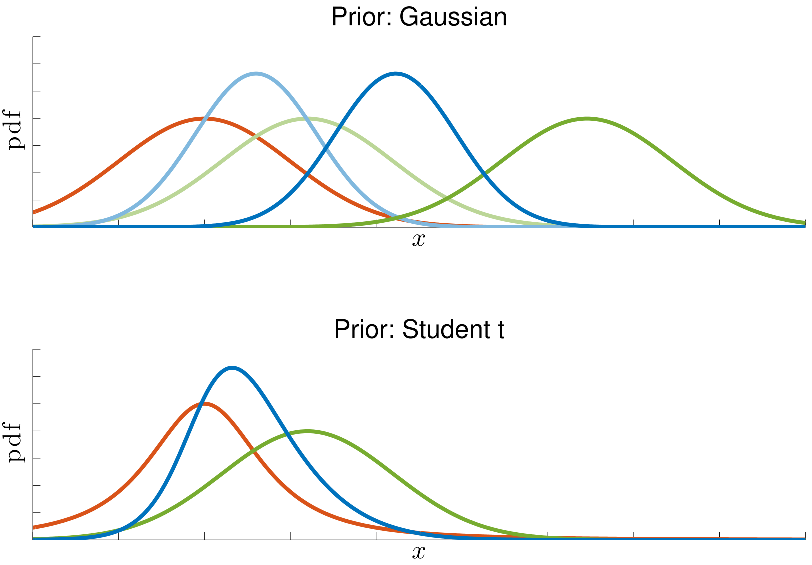
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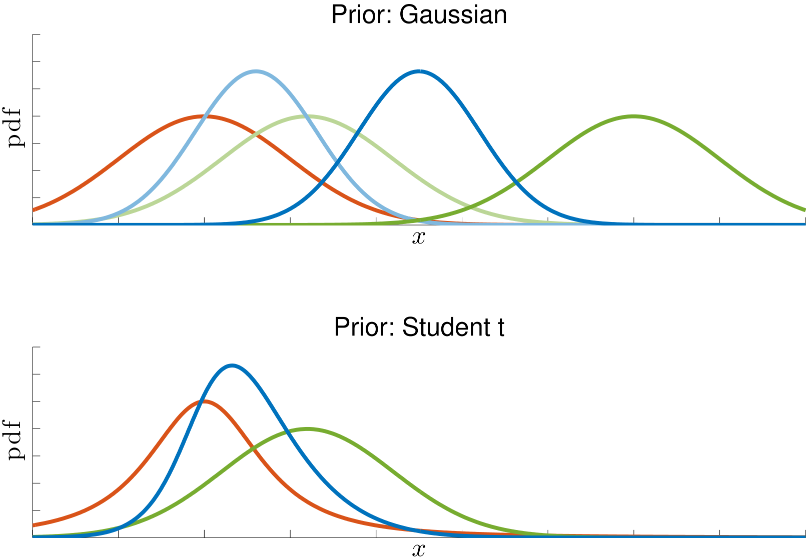
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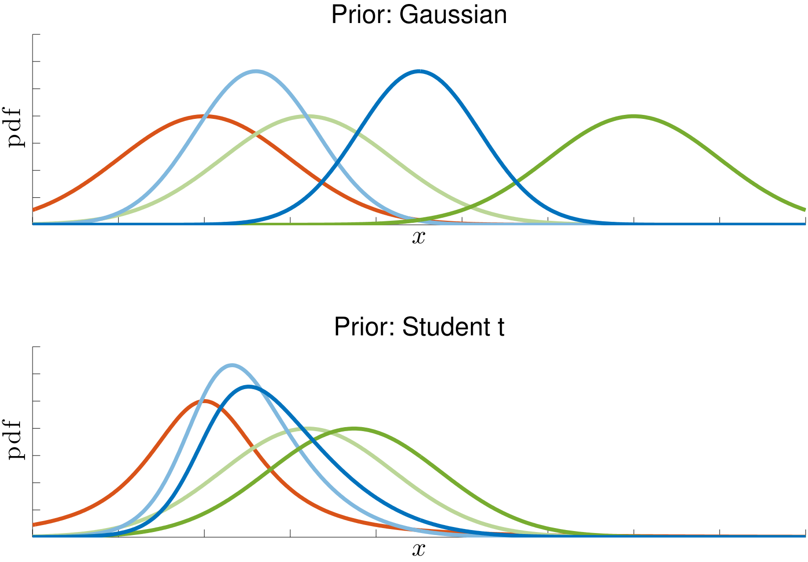
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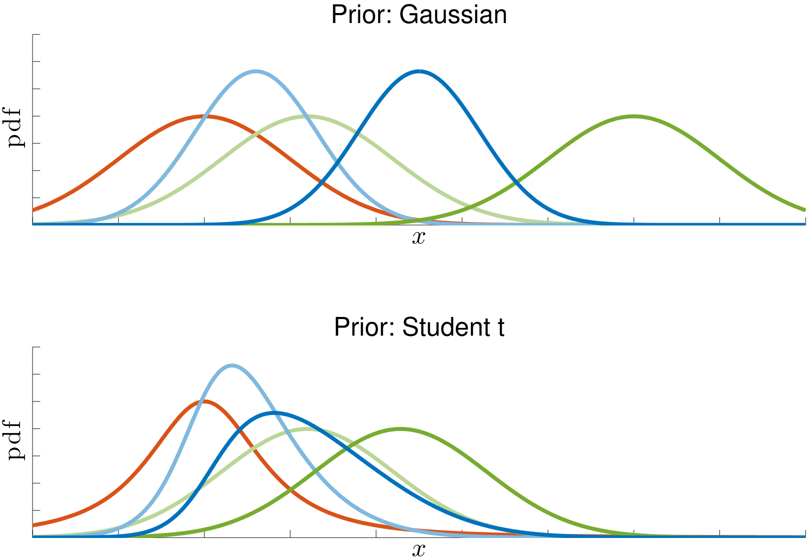

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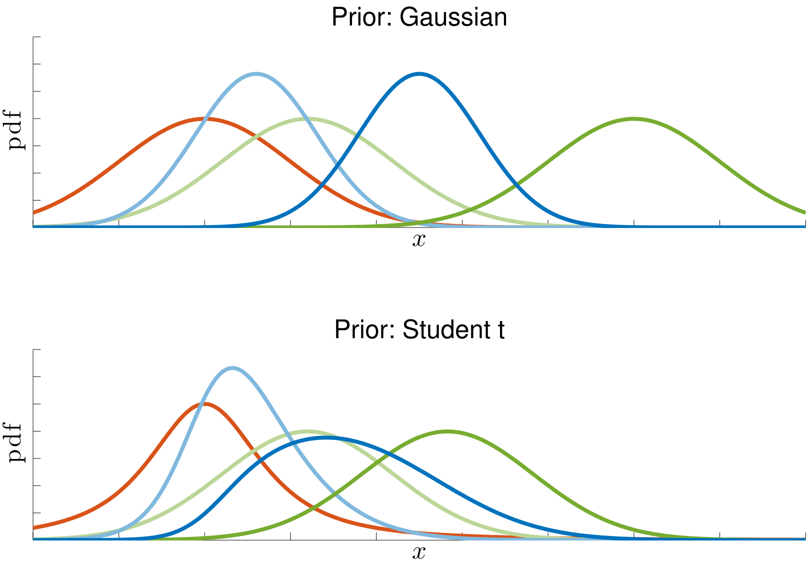
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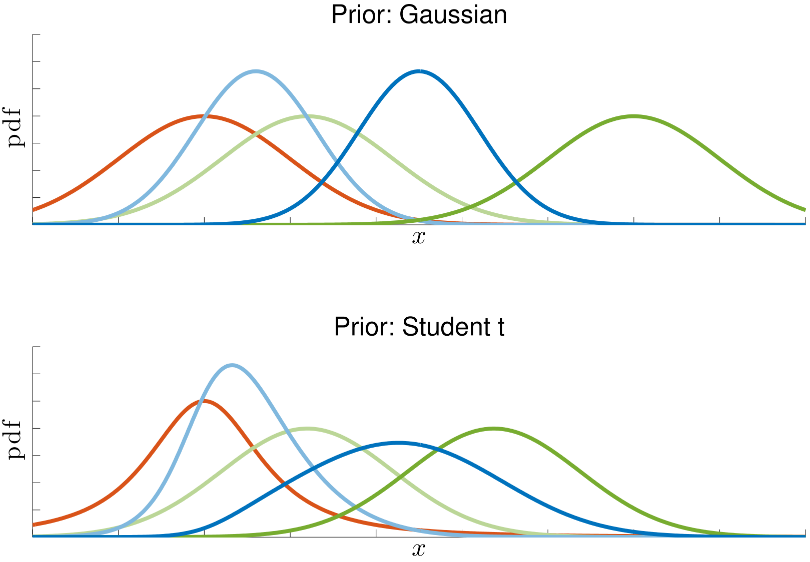
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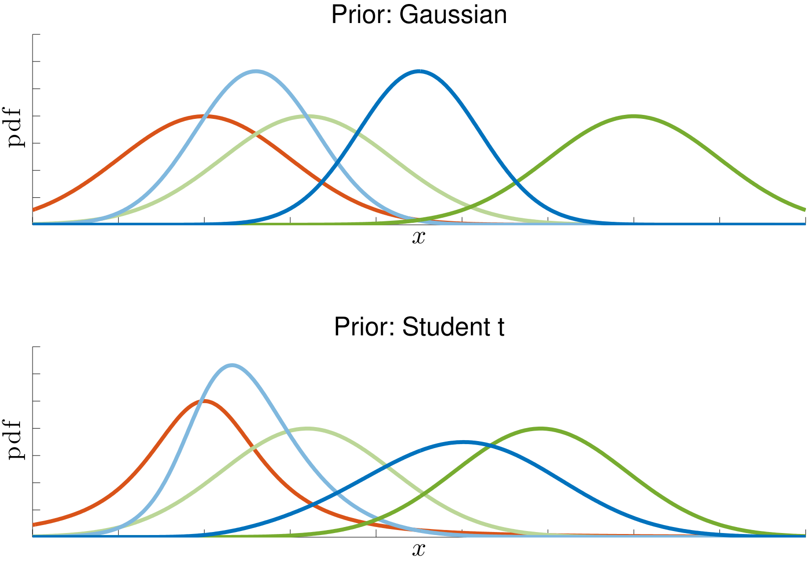
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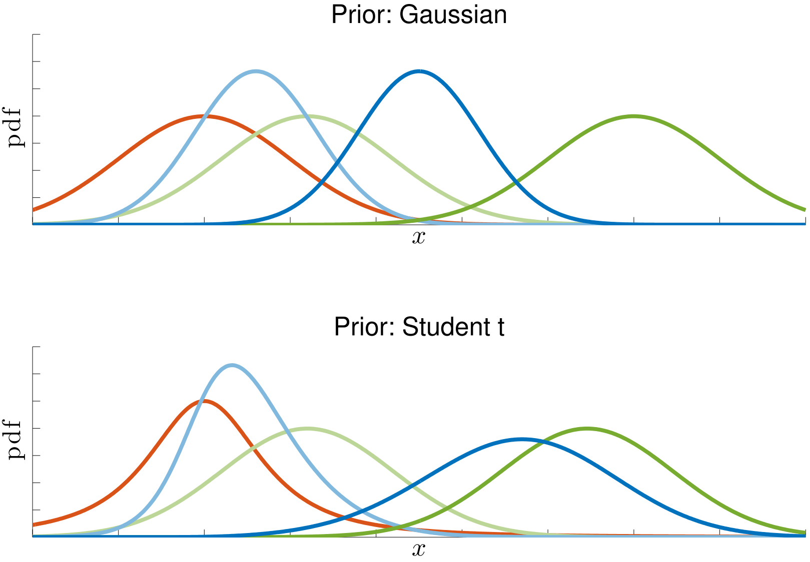
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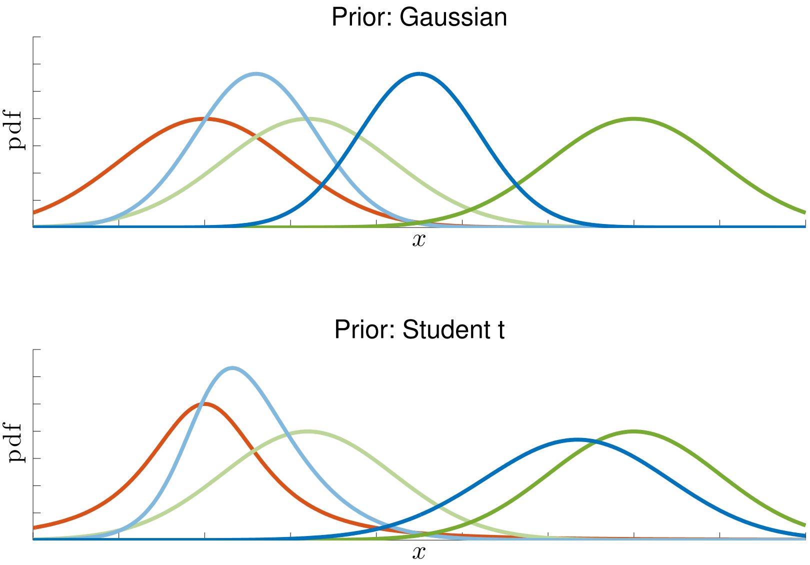
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