# Iterative Regularizing Ensemble Kalman Methods for Inverse Problems

Marco Iglesias

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EnKF workshop, Bergen, Norway

# Nonlinear ill-posed inverse problems

## The forward map

Let $G : X \to Y$ be the forward (parameter-to-observations) map that arises from PDE-constrained problem where $u$ is an unknown parameter (or property)

$$u \longrightarrow G(u)$$

*( $G$ nonlinear, compact, sequentially weakly closed operator between separable Hilbert spaces $X$ and $Y$.)*

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## Example: steady Darcy flow

$$\begin{aligned} -\nabla \cdot e^u \nabla p &= f \quad \text{in } D, \\ -e^u \nabla p \cdot n &= B_N \quad \text{in } \Gamma_N. \\ p &= B_D \quad \text{in } \Gamma_D \end{aligned}$$

where $\partial D = \Gamma_N \cup \Gamma_D$.

$$u = \log(K) \in L^\infty(D) \longrightarrow G(u) = \{p(x_i)\}_{i=1}^N \in \mathbb{R}^M$$

# Nonlinear ill-posed inverse problems

## The data and the noise level

Let $u^\dagger$ be the "truth", i.e. $G(u^\dagger)$ are the exact (noise-free) observations. Assume that we are given *data*

$$y = G(u^\dagger) + \xi^\dagger$$

where $\xi^\dagger$ is noise and we are given a noise level $\eta$ such that

$$||y - G(u^\dagger)|| \leq \eta$$

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## Ill-posedness (Hadamard)

- Existence
- Uniqueness
- Continuity with respect to the data $y$

# Nonlinear ill-posed inverse problems

**Lack of continuity (lack of stability) with respect to the data**

We can construct a sequence $u_n \in X$ such that

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### Regularization

Construct an approximation $u^\eta$ that is stable, i.e. such that

$$u^\eta \to u \qquad \text{as} \qquad \eta \to 0$$

where

$$G(u) = G(u^\dagger)$$

# Regularization

## Regularization Approaches (for nonlinear operators)

- Regularize-then-compute (e.g. Tikhonov, TSVD)
- Compute while regularizing (Iterative Regularization) *[Kaltenbacher, 2010]*
  - *regularizing Levenberg-Marquardt*
  - *Landweber iteration*
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## Aim of this work:

Apply ideas from Iterative Regularization to develop ensemble Kalman methods as *derivative-free* tools for solving nonlinear ill-posed inverse problem in a general abstract framework.

# PDE-constrained Inverse Problems

## The Classical (deterministic) Inverse Problem

Given data $y \in Y$ find

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# PDE-constrained Inverse Problems

## The Classical (deterministic) Inverse Problem

Given data $y \in Y$ find

$$u = \arg\min_{u \in X} ||y - G(u)||^2 \to \min$$

Consider $\mu_0(u) = \mathbb{P}(u)$ the prior on $u$ and

$$y = G(u) + \xi, \qquad \xi \sim \mathcal{N}(0, \Gamma)$$

## The Bayesian Inverse Problem

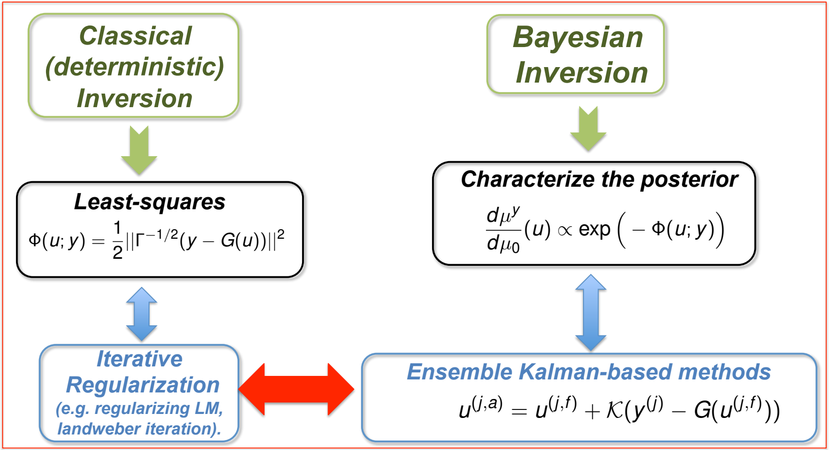
Characterize the posterior $\mu^y(u) = \mathbb{P}(u|y)$:

$$\frac{d\mu^y}{d\mu_0}(u) \propto \exp\Big(-\Phi(u; y)\Big)$$

where

$$\Phi(u; y) = \frac{1}{2}||\Gamma^{-1/2}(y - G(u))||^2$$

# Overview of this work



# Reference

Iterative Regularization for Data Assimilation in Petroleum Reservoirs

*Multiscale Inverse Problems Workshop, Warwick University, June 17-19, 2013.*

http://www2.warwick.ac.uk/fac/sci/maths/research/events/2012-2013/nonsymp/mip/schedule/

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## Iterative regularization for ensemble data assimilation in reservoir models

Marco A. Iglesias

*(Submitted on 21 Jan 2014)*

We propose the application of iterative regularization for the development of ensemble methods for solving Bayesian inverse problems. In concrete, we construct (i) a variational iterative regularizing ensemble Levenberg-Marquardt method (IR-enLM) and (ii) a derivative-free iterative ensemble Kalman smoother (IR-ES). The aim of these methods is to provide a robust ensemble approximation of the Bayesian posterior. The proposed methods are based on fundamental ideas from iterative regularization methods that have been widely used for the solution of deterministic inverse problems [21]. In this work we are interested in the application of the proposed ensemble methods for the solution of Bayesian inverse problems that arise in reservoir modeling applications. The proposed ensemble methods use key aspects of the regularizing Levenberg-Marquardt scheme developed by Hanke [16] and that we recently applied for history matching in [18].

In the case where the forward operator is linear and the prior is Gaussian, we show that the proposed IR-enLM and IR-ES coincide with standard randomized maximum likelihood (RML) and the ensemble smoother (ES) respectively. For the general nonlinear case, we develop a numerical framework to assess the performance of the proposed ensemble methods at capturing

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# Iterative ensemble Kalman "Smoother"

Assume that the data $y = G(u^\dagger) + \xi$ with $\xi \sim N(0, \Gamma)$.

Consider an initial ensemble $u_0^{(1)}, \dots, u_0^{(N_e)}$.

**Prediction** $u_n^{(j)} \to G(u_n^{(j)})$

$$\overline{u}_n = \frac{1}{N_e}\sum_{j=1}^{N_e} u_n^{(j)}, \qquad \overline{w}_n = \frac{1}{N_e}\sum_{j=1}^{N_e} G(u_n^{(j)})$$

$$C^{ww} = \frac{1}{N_e - 1}\sum_{j=1}^{N_e}(G(u_n^{(j)}) - \overline{w}_n)(G(u_n^{(j)}) - \overline{w}_n)^T,$$

$$C^{uw} = \frac{1}{N_e - 1}\sum_{j=1}^{N_e}(u_n^{(j)} - \overline{u}_n)(G(u_n^{(j)}) - \overline{w}_n)^T$$

**Analysis** $u_n^{(j)} \to u_{n+1}^{(j)}$

$$u_{n+1}^{(j)} = u_n^{(j)} + C^{uw}(C^{ww} + \Gamma)^{-1}(y^{(j)} - G(u_n^{(j)})), \qquad y^{(j)} = y + \eta^{(j)}$$

# Ensemble Kalman Methods for Inverse Problems

## Augmented analysis

$$
\begin{aligned}
u_{n+1}^{(j)} &= u_n^{(j)} + C^{uw}(C^{ww} + \Gamma)^{-1}(y^{(j)} - G(u_n^{(j)})) \\
w_{n+1}^{(j)} &= G(u_n^{(j)}) + C^{ww}(C^{ww} + \Gamma)^{-1}(y^{(j)} - G(u_n^{(j)}))
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z_{n+1}^{(j)} = z_n^{(j)} + C^f H^T \left(HC^f H^T + \Gamma\right)^{-1}(y^{(j)} - Hz_n^{(j)})
$$

where

$$
z = (u,w)^T \in Z \equiv X \times Y \qquad H = (0, I) \qquad C^f = \begin{pmatrix} C^{uu} & C^{uw} \\ (C^{uw})^T & C^{ww} \end{pmatrix}
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## Kalman as a Tikhonov-regularized linear inverse problems

$$
\overline{z}_{n+1} = \frac{1}{N_e}\sum_{j=1}^{N_e} z_{n+1}^{(j)} = \mathrm{argmin}_z \left( ||\Gamma^{-\frac{1}{2}}(y - Hz)||^2 + ||(C^f)^{-\frac{1}{2}}(z - \overline{z}_n)||_Z^2 \right)
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where $\overline{z}_n \equiv \frac{1}{N_e}\sum_{j=1}^{N_e} z_n^{(j)}$.

**Kalman as a Tikhonov-regularized linear inverse problems**

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In summary, this iterative method is solving a sequence of linear inverse problems: Given $y$, find $z$ such that

$$y = Hz$$

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**Preliminary work on Kalman methods for inverse problems**


M. Iglesias, K. Law and A.M. Stuart,

Ensemble Kalman methods for inverse problems. *Inverse Problems.* 29 (2013) 045001 http://arxiv.org/abs/1209.2736

# Ensemble Kalman Methods for Inverse Problems

- This algorithm can be formulated in a general abstract framework on Hilbert spaces.
- No localization/inflation/truncation of any type was considered. The main focus was the initial ensemble.
- We consider both initial ensemble sample from the prior but also the first elements of a basis.
- Invariance subspace property $\mathcal{A} = \mathrm{span}\{u_0^{(j)}\}_{j=1}^{N_e}$.

**Theorem (Iglesias, Law, Stuart)**

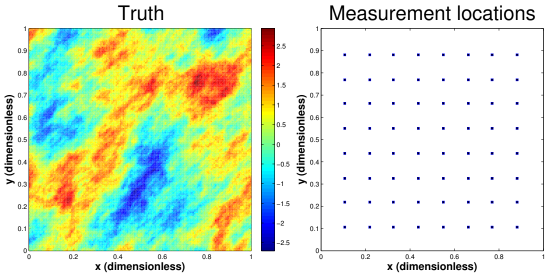
$$\overline{u}_n^{(j)} \in \mathcal{A} \text{ for all } (n, j) \in \mathbb{N} \times \{1, \cdots, N_e\}.$$

- Even though the problem is defined on a compact subspace $\mathcal{A}$, we found that further regularization is needed.

# Synthetic experiment with a toy (groundwater) model

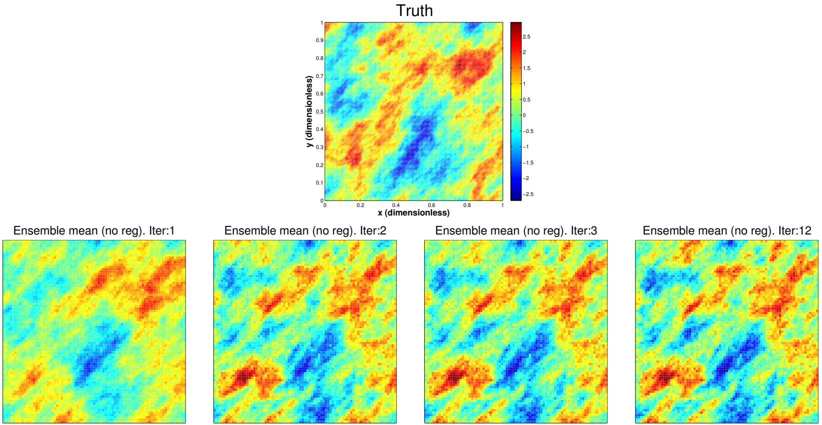
Consider

- Initial ensemble generated from a prior $\mathbb{P}(u) = N(\overline{u}, C)$.
- Let $G(u)$ be the forward operator that arises from a reservoir model.




Consider a truth $u^{\dagger} \sim \mathbb{P}(u)$ from which synthetic data are generated by $y = G(u^{\dagger}) + \eta \qquad \eta \sim N(0, \Gamma)$ (prescribed $\Gamma$ covariance of the Gaussian noise).

# Reconstructing the truth with the mean of an ensemble of $N_e = 75$ (with small noise)

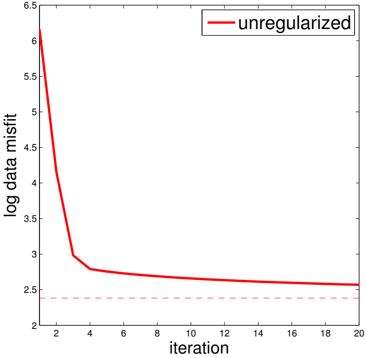


# Performance

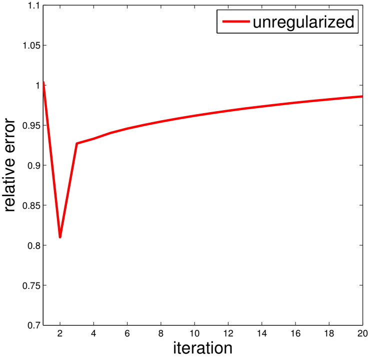
$$\overline{u}_n \equiv \frac{1}{N}\sum_{j=1}^{N} u_n^{(j)}$$

$||\Gamma^{-1/2}(y - \overline{u}_n)||_{l^2}$ $\qquad$ $||\overline{u}_n - u^{\dagger}||_{L^2(D)}$

*Data misfit*




*Error w.r.t truth*



Recall the standard update formula yields a mean defined by

$$\overline{z}_{n+1} = \operatorname{argmin}_z \left( ||\Gamma^{-\frac{1}{2}}(y - Hz)||^2 + ||(C^f)^{-\frac{1}{2}}(z - \overline{z}_n)||_Z^2 \right)$$

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**We propose in**

M. A. Iglesias
Iterative regularization for ensemble-based data assimilation in reservoir models.
*In review (http://arxiv.org/abs/1401.5375).* 2014 (Submitted to Computational Geosciences)

to modify the ensemble method

$$z_{n+1}^{(j)} = z_n^{(j)} + C^f H^T \left( HC^f H^T + \alpha\Gamma \right)^{-1} (y^{(j)} - Hz_n^{(j)})$$

so that the mean is given by

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i.e. Regularizing Kalman as in Levenberg-Marquardt!! where a selection of $\alpha$ guided by Iterative Regularization methods. More precisely, we propose

$$\rho||\Gamma^{-1/2}(y - H\overline{z}_n)|| \leq ||\Gamma^{-1/2}(y - H\overline{z}_{n+1}(\alpha))|| \leq ||\Gamma^{-1/2}(y - H\overline{z}_n)||$$

where $\rho$ is a tunable parameter in $(0, 1)$.

**Proposition [Iglesias 2014]**

$$\alpha \rightarrow ||\Gamma^{-1/2}(y - H\overline{z}_{n+1}(\alpha))||$$

is continuous and monotone nondecreasing. Moreover,

- $\lim_{\alpha\rightarrow 0} ||\Gamma^{-1/2}(y - H\overline{z}_{n+1}(\alpha))|| = ||\Gamma^{-1/2}(y - H\overline{z}_{n})||$
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Thus, there exists $\alpha$ such that

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**Stopping criteria: *Discrepancy Principle***

We terminate the algorithm whenever

$$||\Gamma^{-1/2}(y - H\overline{z}_{n})|| \leq \tau\eta||\Gamma^{-1/2}||$$

for a prescribed value $\tau > 1/\rho$ where $\eta = ||y - G(u^{\dagger})||$ is the noise level.

Then we can show that the selection of $\alpha$ is consistent with the discrepancy principle to the linear inverse problem corresponding to the analysis step.

## An iterative regularizing ensemble Kalman method

Let $\rho < 1$ and $\tau > 1/\rho$. Generate an initial ensemble $u_0^{(j)} \sim \mu_0$

### A regularizing Kalman method

**(1) Prediction Step:** Evaluate $w_m^{(j,f)} = G(u_m^{(j)})$ define $\overline{w}_m^f$

**(2)** Stopping criteria. If

$$||\Gamma^{-1/2}(y - \overline{w}_m^f)|| \leq \tau\eta$$

Stop. Otherwise: define $C_m^{uw}$, $\overline{u}_m$, $C_m^{ww}$ and

**(3) Analysis step:** Compute the updated ensembles

$$u_{m+1}^{(j)} = u_m^{(j)} + C_m^{uw}(C_m^{ww} + \alpha_m\Gamma)^{-1}(y^{(j)} - w_m^{(j,f)})$$

for $\alpha_m$ such that

$$\alpha_m||\Gamma^{1/2}(C_m^{ww} + \alpha_m\Gamma)^{-1}(y^\eta - \overline{w}_m^f)|| \leq \rho||\Gamma^{-1/2}(y^\eta - \overline{w}_m^f)||$$

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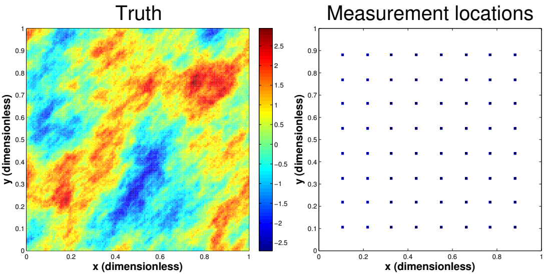


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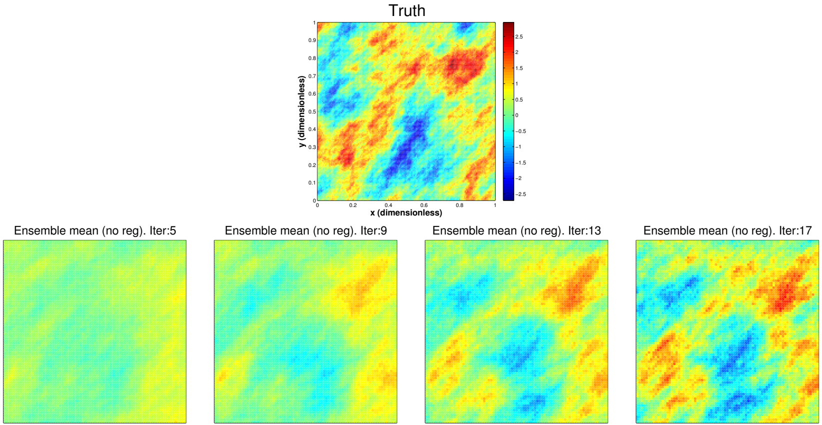

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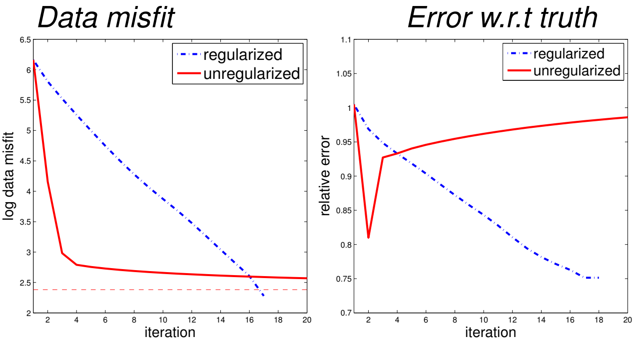
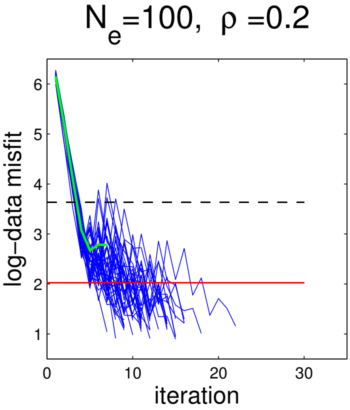
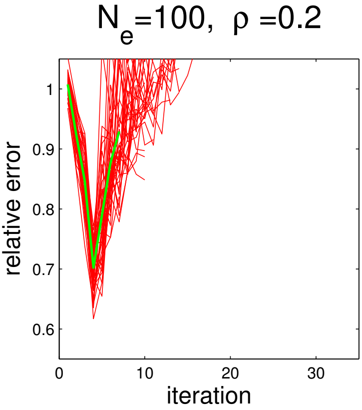


# Regularizing properties as a function of $\rho$



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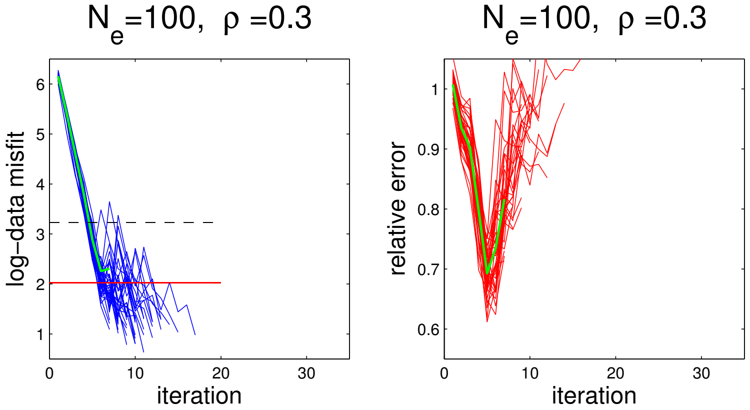

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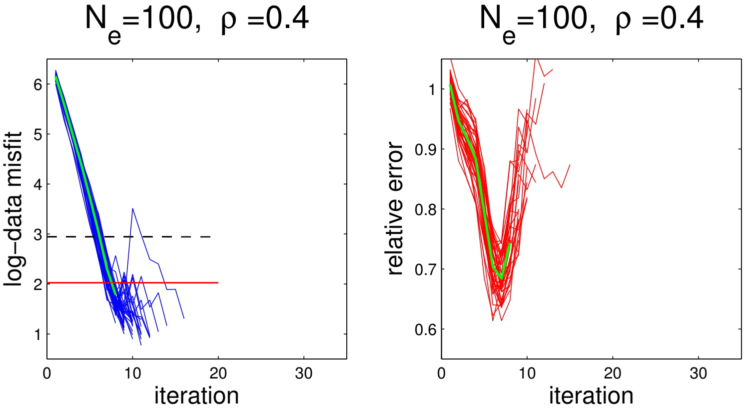

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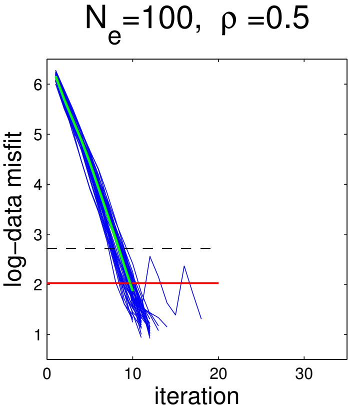

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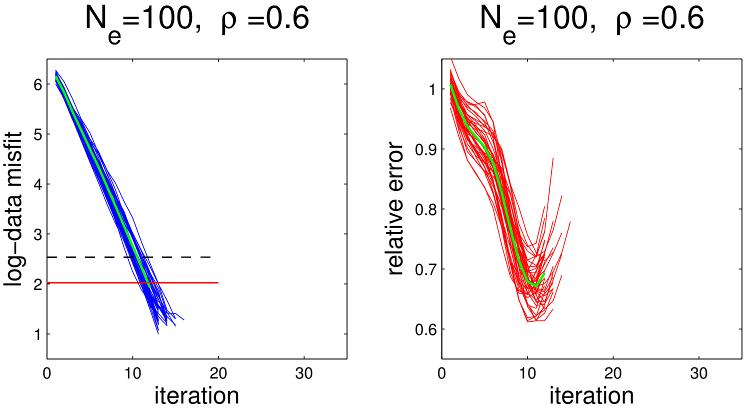

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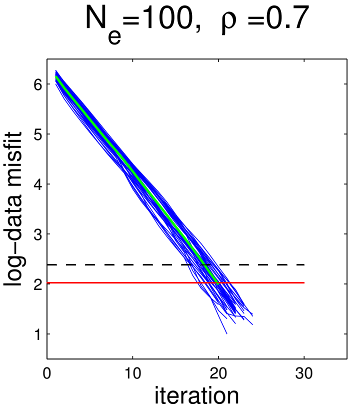


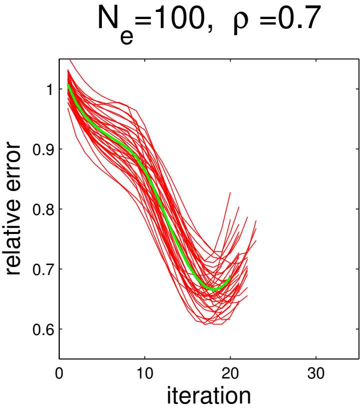

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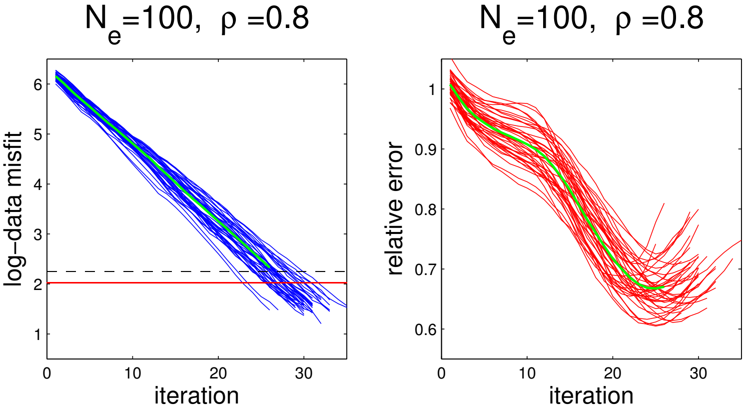
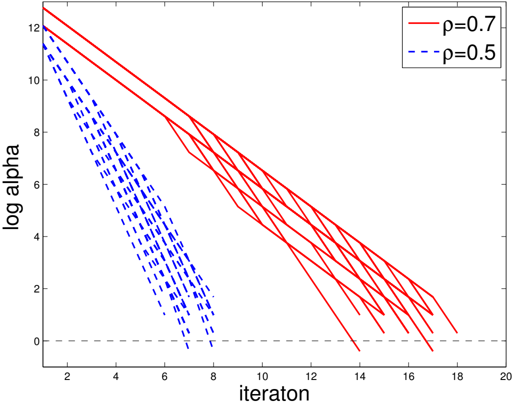
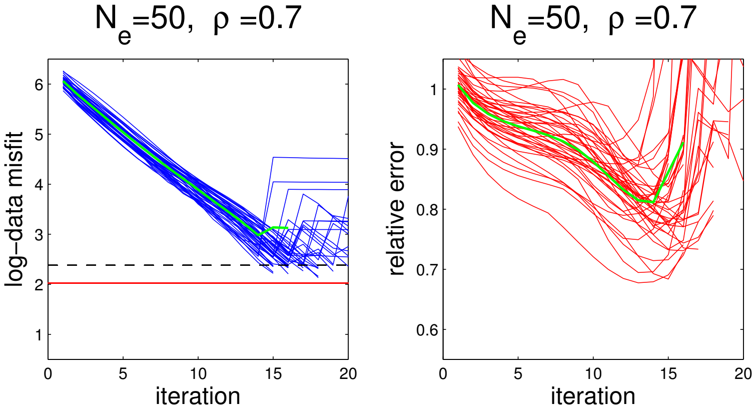


# Regularization parameter $\alpha$

Plot of $\log\alpha$



# Regularizing properties as a function of the ensemble size



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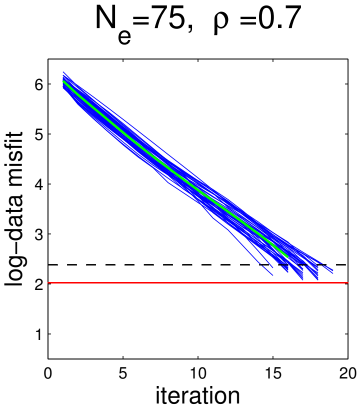


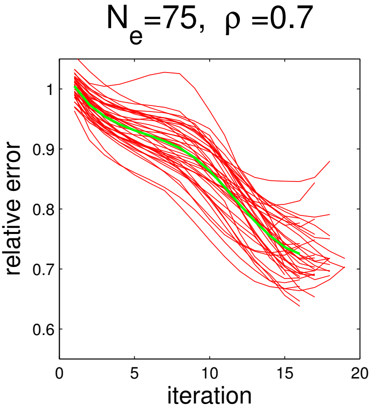

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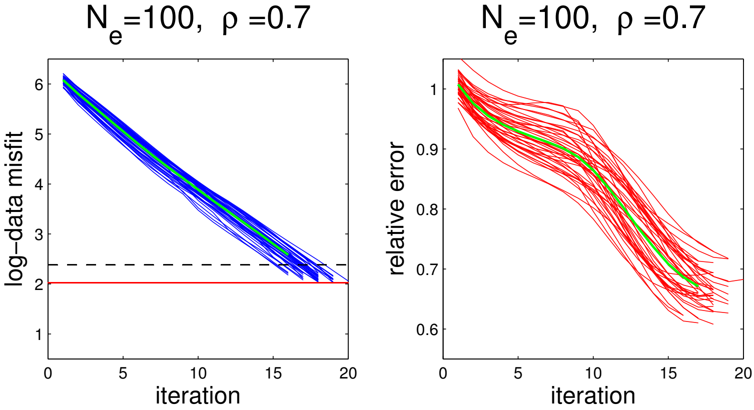

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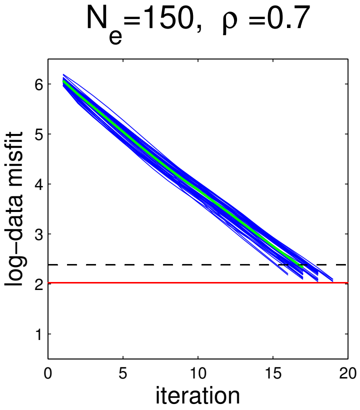


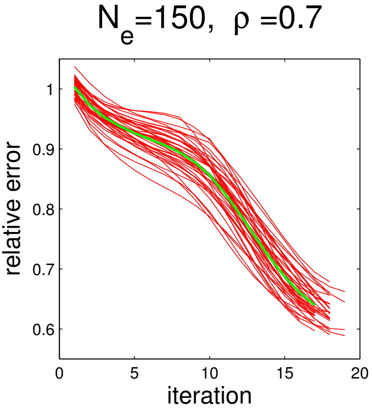

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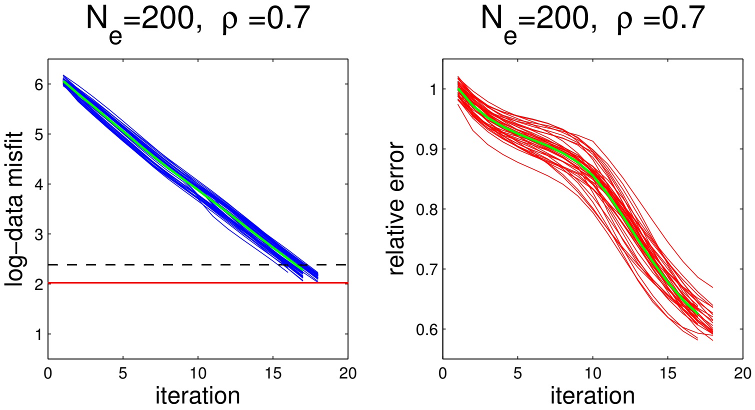

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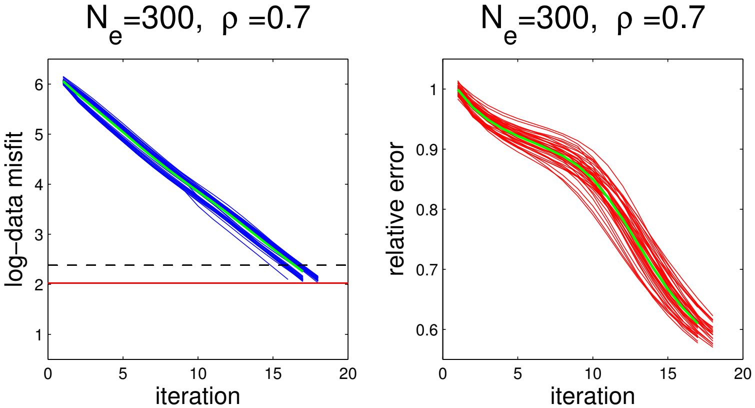
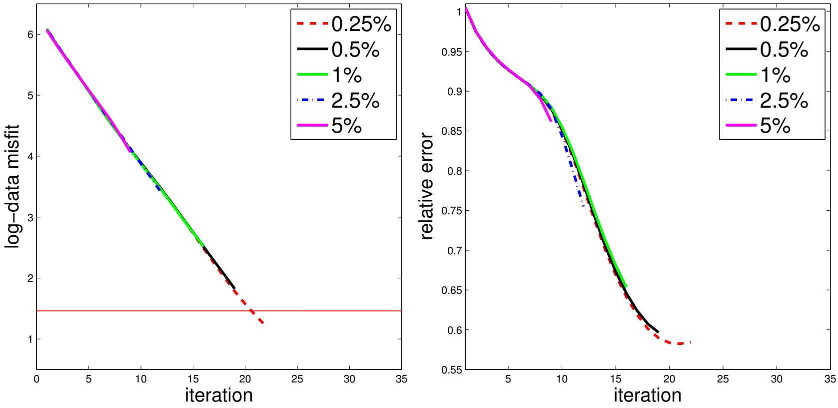


# Convergence as the noise level decreases

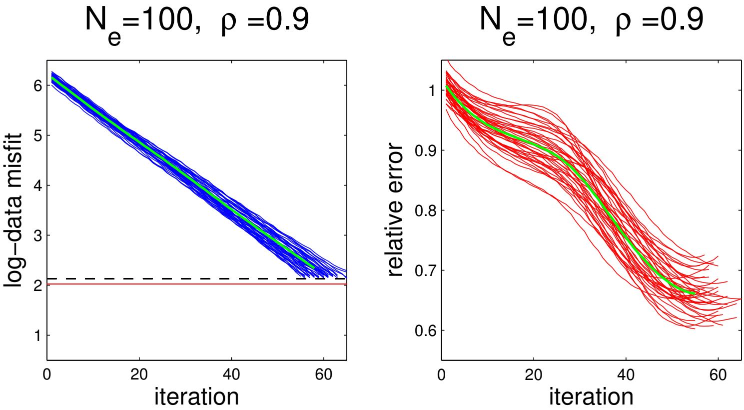
$$\overline{u}_n \equiv \frac{1}{N}\sum_{j=1}^{N} u_n^{(j)}$$

$$||\Gamma^{-1/2}(y - \overline{u}_n)||_{l^2} \qquad\qquad ||\overline{u}_n - u^{\dagger}||_{L^2(D)}$$



# Computational cost

Cost approx 6000!!!



# Accelerating EnKF

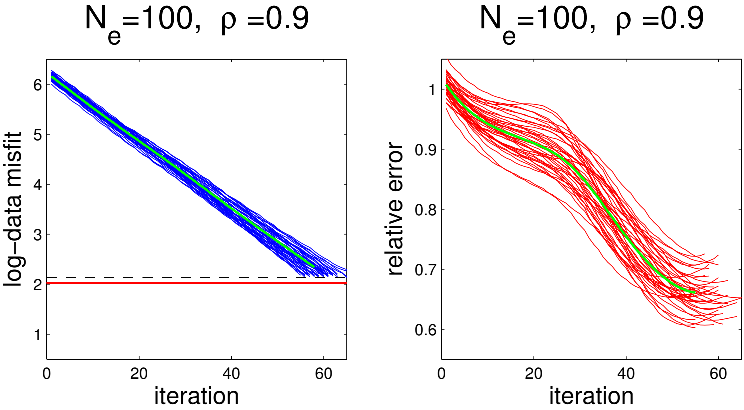
Recall the augmented analysis

$$\begin{aligned} u_{n+1}^{(j)} &= u_n^{(j)} + C^{uw}(C^{ww} + \Gamma)^{-1}(y^{(j)} - G(u_n^{(j)})) \\ w_{n+1}^{(j)} &= G(u_n^{(j)}) + C^{ww}(C^{ww} + \Gamma)^{-1}(y^{(j)} - G(u_n^{(j)})) \end{aligned}$$

Use the second equation to do some linear iterations when $\rho$ is large (and so is $\alpha$). adhoc!!

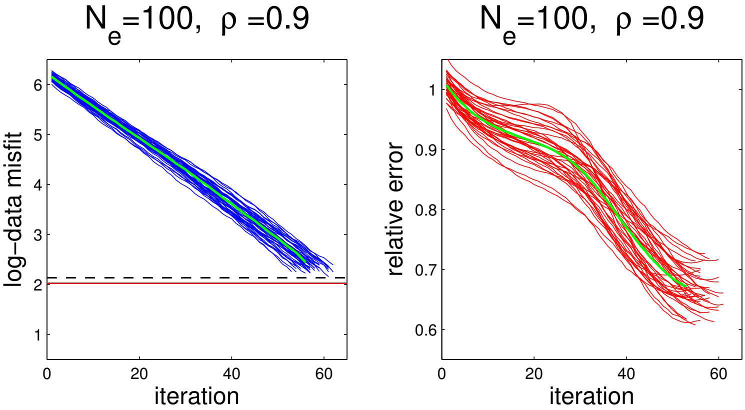
# Computational cost

1 nonlinear iteration every iteration
Cost approx 6000!!!



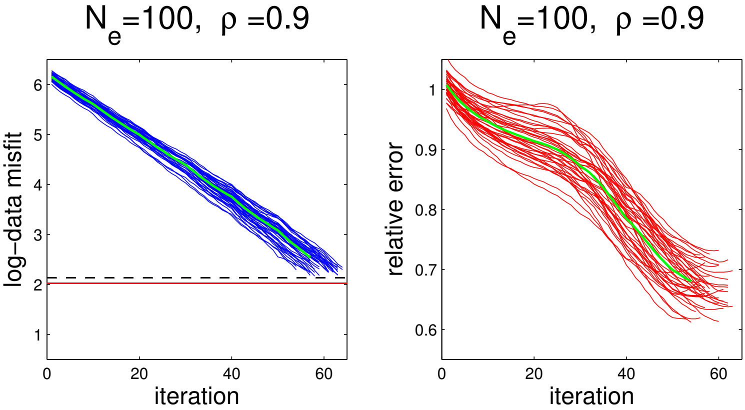
# Computational cost

1 nonlinear iteration every 5
Cost approx 1200!!!



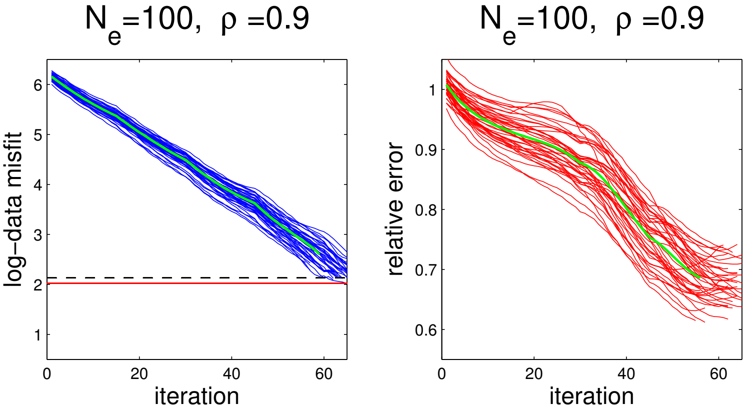
# Computational cost

1 nonlinear iteration every 10
Cost approx 600!!!



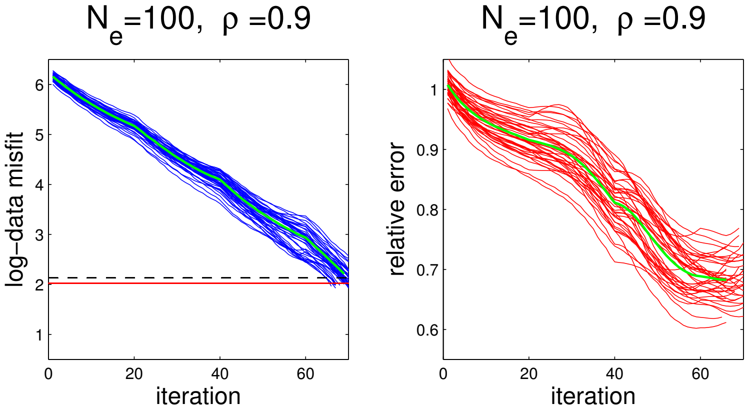
# Computational cost

1 nonlinear iteration every 15
Cost approx 400!!!



# Computational cost

1 nonlinear iteration every 20
Cost approx 300!!!



# Connections with variational Iterative Regularization

**Assume that at a given iteration level**

$$G(u_m^{(j)}) \approx G(\overline{u}_m) + DG(\overline{u}_m)(u_m^{(j)} - \overline{u}_m)$$

The update formula becomes

$$\overline{u}_{m+1} = \overline{u}_m + C_m^{uu} DG(\overline{u}_m)^* (DG(\overline{u}_m) C_m^{uu} DG(\overline{u}_m)^* + \alpha_m \Gamma)^{-1} (y - G(\overline{u}_m))$$

where

$$C_m^{uu} = \frac{1}{N_e - 1} \sum_{j=1}^{N_e} (u_m^{(j,f)} - \overline{u}_m^f)(u_m^{(j,f)} - \overline{u}_m^f)^T$$

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**Assume that at a given iteration level**

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where

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If we replace $C_m^{uu}$ by the prior error covariance $C$, then

$$\overline{u}_{m+1} = \overline{u}_m + CDG(\overline{u}_m)^* (DG(\overline{u}_m) CDG(\overline{u}_m)^* + \alpha_m \Gamma)^{-1} (y - G(\overline{u}_m))$$

This is Levenberg-Marquardt applied for the minimization

$$u = \arg\min_{u \in X} ||\Gamma^{-1/2}(y - G(u))||^2 \rightarrow \min$$

in $X$ with norm $||C^{-1/2} \cdot ||_X$. (No regularization term!!!!)

# The regularizing LM

Selecting $\alpha_m$ and the stopping criteria according to the discrepancy principle yields the regularizing Levenberg-Marquardt of [Hanke, 1997]:

## Theorem [Hanke 1997]

$\overline{u}_m$ converges after a finite number of iterations and

$$\overline{u}_m \to u \qquad \text{as} \qquad \eta \to 0 \qquad (\text{where } \; G(u) = G(u^\dagger))$$

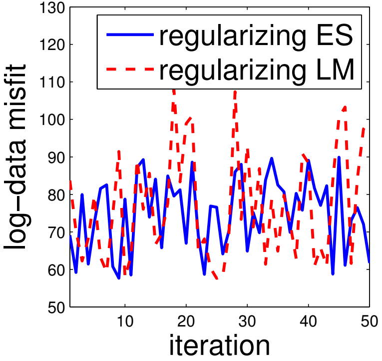
## The regularizing LM scheme for reservoir modeling applications

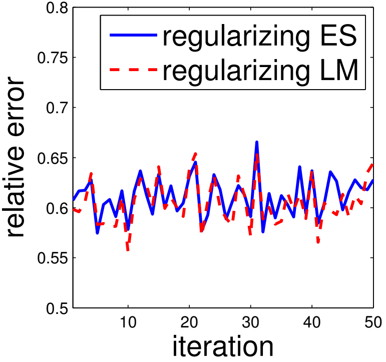

M. A. Iglesias
Iterative regularization for ensemble-based data assimilation in reservoir models.
*In review (http://arxiv.org/abs/1401.5375).* 2014

M. A. Iglesias and C. Dawson
The regularizing Levenberg-Marquardt scheme for history matching of petroleum reservoirs, *Computational Geosciences*, (2013) 17:1033-1053

## The proposed ES as an approximate regularizing LM scheme

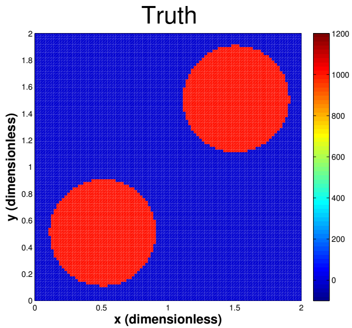
Comparing ES with the regularizing LM scheme (on the same subspace)



## Ensemble Kalman method for geometric inverse problems

Suppose we are interested in recovering something like:




We parametrize the permeability in terms of a level set function $u$. i.e.

$$K(u) = K_i \mathbb{1}_{u<0} + K_e \mathbb{1}_{u\geq 0}$$

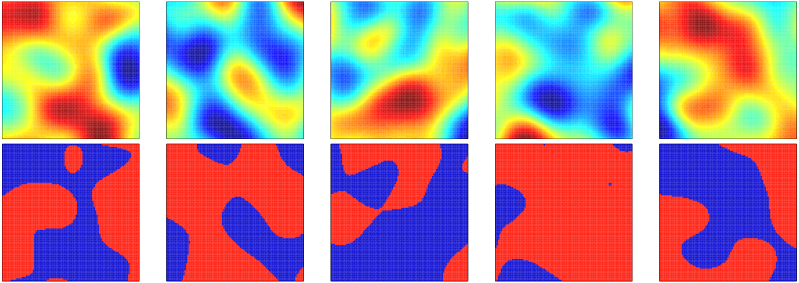
$p = G(u)$ is, as before, the solution to $-\nabla \cdot K(u)\nabla p = f$ evaluated at some locations. We invert noisy measurements: $y = G(u) + \xi$

## Ensemble Kalman method for geometric inverse problems

Let us consider an artificial prior (for the initial ensemble)

$$\mu_0 = N(0, C)$$

with some covariance that reflects the regularity of the shape.



$$K(u) = K_i \mathbb{1}_{u<0} + K_e \mathbb{1}_{u\geq 0}$$

# Ensemble Kalman method for geometric inverse problems

# Summary

- Iterative regularization provides strategies for regularizing Kalman based methods.
- Regularization has strong effect in the robustness and accuracy of ensemble methods for solving both classical and Bayesian inverse problems.
- Further investigations are required to establish the mathematical properties of these approximations.

## References


M. A. Iglesias

Iterative regularization for ensemble-based data assimilation in reservoir models. *In review (http://arxiv.org/abs/1401.5375).* 2014

M. Iglesias, K. Law and A.M. Stuart,

Ensemble Kalman methods for inverse problems. *Inverse Problems.* 29 (2013) 045001 http://arxiv.org/abs/1209.2736

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The regularizing Levenberg-Marquardt scheme for history matching of petroleum reservoirs, *Computational Geosciences*, (2013) 17:1033-1053